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A TEXTBOOK

ON

SURVEYING AND MAPPING

INTERNATIONAL CORRESPONDENCE SCHOOLS
SCRANTON, PA.

ARITHMETIC
FORMULAS
GEOMETRY AND TRIGONOMETRY
SURVEYING
LAND SURVEYING
MAPPING

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PREFACE

The Instruction Papers which are furnished to the students of the International Correspondence Schools become so badly worn and soiled when a student has completed his Course, that he has worn them out, and they are no longer suitable for reference or review. Since the Instruction Papers are very valuable, especially to those who have studied them, there grew up a demand for sets of the Instruction Papers, indexed for convenient reference, and durably bound for preservation. To meet this demand the Papers were revised, reset, and electrotyped, and are herewith presented permanently bound in handsome half-leather volumes.

The volumes for the present Course, the Surveying and Mapping, are three in number:

The first volume in the order of study contains the Instruction Papers and Examination Questions on Arithmetic, Formulas, Geometry and Trigonometry, Surveying, Land Surveying, and Mapping.

Of the two additional volumes, one volume contains the Instruction Paper on Geometrical Drawing, and the plates that go with it, and the instructions for drawing them. On account of the large size of this volume in former editions and an increasing demand on the part of students that this volume correspond in size with the others of the set, we have had it reset and have inserted the plates on guards. As both this volume and the one containing the Tables and

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Formulas were very thin, we have combined them under one cover, thus reducing the number of volumes from four to three.

The other volume contains the answers to the Examination Questions.

The Instruction Papers are written from a practical standpoint, and contain only such information as the student requires in order to obtain a good working knowledge of the subjects pertaining to Surveying and Mapping.

In order that the student may make rapid progress, we do not enter into speculative discussions of different subjects; neither do we give him several ways of arriving at the same result, for this tends to confuse the student and to leave him in doubt as to which is best suited to his purpose. We give him the best or most suitable rule, formula, or method that we know of, without mentioning any others. In short, the papers are written for practical men, and all the difficulties encountered by a student studying by himself, particularly those which are due to a definition or explanation being too technical, abstract, or not clear enough to be readily understood, have been carefully considered and overcome.

INTERNATIONAL CORRESPONDENCE SCHOOLS.

NOTICE.

There is a break in the continuity of the article numbers, figure numbers, etc., between the section on Formulas and the section on Geometry and Trigonometry, and between Geometry and Trigonometry and Surveying. This does not affect the subject-matter, which is published in full, and in regular order.

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UNIV. OF CALIFORNIA

ARITHMETIC.

(SECTION 1.)

DEFINITIONS.

1. Arithmetic is the art of reckoning, or the study of numbers.

2. A **unit** is *one*, or a single thing, as *one*, *one* boy, *one* horse, *one* dozen.

3. A **number** is a unit or a collection of units, as *one*, *three* apples, *five* boys.

4. The **unit of a number** is one of the collection of units which constitutes the number. Thus, the unit of *twelve* is *one*, of *twenty* dollars is *one* dollar.

5. A **concrete number** is a number applied to some particular kind of object or quantity, as three *horses*, five *dollars*, ten *pounds.*

6. An **abstract number** is a number that is not applied to any object or quantity, as *three*, *five*, *ten.*

7. Like numbers are numbers which express units of the *same kind*, as 6 *days* and 10 *days*, 2 *feet* and 5 *feet.*

8. Unlike numbers are numbers which express units of *different kinds*, as ten *months* and eight *miles*, seven *dollars* and five *feet.*

NOTATION AND NUMERATION.

9. Numbers are expressed in three ways: (1) by words; (2) by figures; (3) by letters.

10. Notation is the art of expressing numbers by figures or letters.

11. Numeration is the art of reading the numbers which have been expressed by figures or letters.

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12. The **Arabic notation** is the method of expressing numbers by figures. This method employs ten different **figures** to represent numbers, viz.:

Figures	0	1	2	3	4	5	6	7	8	9
Names	*naught, cipher, or zero.*	*one*	*two*	*three*	*four*	*five*	*six*	*seven*	*eight*	*nine*

The first character (0) is called **naught, cipher,** or **zero,** and, when standing alone, has no value.

The other nine figures are called **digits,** and each one has a value of its own.

Any whole number is called an **integer.**

13. As there are only ten *figures* used in expressing numbers, each *figure* must express a different *value* at different times.

14. The value of a figure depends upon its *position* in relation to others.

15. Figures have **simple** values and **local** or **relative** values.

16. The **simple** value of a figure is the value it expresses when standing alone.

17. The **local** or **relative** value is the *increased* value it expresses by having other figures placed on its right.

For instance, if we see the figure 6 standing alone, thus. 6
we consider it as *six units*, or simply **six.**

Place another 6 to the *left* of it; thus. 66
The original figure is still *six units*, but the second one is *ten times* 6, or 6 **tens.**

If a third 6 be now placed still one place further to the *left*, it is increased in value *ten times* more, thus making it 6 **hundreds** 666

A fourth 6 would be 6 **thousands** 6666

A fifth 6 would be 6 **tens of thousands,** or **sixty thousand** 66666

A sixth 6 would be 6 **hundreds of thousands** . 666666

A seventh 6 would be 6 **millions** 6666666

The entire line of seven figures is read *six millions, six hundred sixty-six thousands, six hundred sixty-six.*

18. The **increased value** of each of these figures is its *local*, or *relative*, value. Each figure is *ten times* greater in value than the one immediately on its *right*.

19. The **cipher** (0) has no value itself, but it is useful in determining the place of other figures. To represent the number *four hundred five*, two digits only are necessary, one to represent *four hundred* and the other to represent *five units;* but if these two digits are placed together, as 45, the 4 (being in the second place) will mean 4 *tens*. To mean 4 *hundreds*, the 4 should have two figures on its right, and a *cipher* is therefore inserted in the place usually given to *tens*, to show that the number is composed of *hundreds* and *units* only, and that there are no *tens*. *Four hundred five* is therefore expressed as 405. If the number were *four thousand five*, two ciphers would be inserted; thus, 4005. If it were *four hundred fifty*, it would have the *cipher* at the right-hand side to show that there were no *units*, and only *hundreds* and *tens;* thus, 450. *Four thousand fifty* would be expressed 4050, the first cipher indicating that there are no hundreds and the second that there are no units.

NOTE.—When speaking of the figures of a number by referring to them as first figure, second figure, etc., always begin to count at the *left*. Thus, in the number 41,625, 4 is the first figure, 6 the third figure, 5 the fifth or last figure, etc.

20. In *reading* figures, it is usual to point off the number into groups of three figures each, beginning with the right-hand or **units** column, a comma (,) being used to point off these groups.

Billions.			*Millions.*			*Thousands.*			*Units.*		
Hundreds of Billions.	**Tens of Billions.**	**Billions.**	**Hundreds of Millions.**	**Tens of Millions.**	**Millions.**	**Hundreds of Thousands.**	**Tens of Thousands.**	**Thousands.**	**Hundreds of Units.**	**Tens of Units.**	**Units.**
4	**3**	**2**,	**1**	**9**	**8**,	**7**	**6**	**5**,	**4**	**3**	**2**

In *pointing off* these figures, begin at the right-hand figure and count—*units*, *tens*, *hundreds;* the next group of three figures is *thousands*, therefore, we insert a comma (,) before beginning with them. Beginning at the figure 5, we say *thousands*, *tens of thousands*, *hundreds of thousands*, and insert another comma; we next read *millions*, *tens of millions*, *hundreds of millions*, and insert another comma; we then read *billions*, *tens of billions*, *hundreds of billions.*

The entire line of figures would be read: *Four hundred thirty-two billions*, *one hundred ninety-eight millions*, *seven hundred sixty-five thousands*, *four hundred thirty-two.* When we thus *read* a line of figures it is called **numeration,** and if the **numeration** be changed back to *figures*, it is called **notation.**

For instance, the writing of the figures,

72,584,623,

would be the **notation,** and the **numeration** would be *seventy-two millions*, *five hundred eighty-four thousands*, *six hundred twenty-three.*

21. NOTE.—It is customary to leave the *s* off the words millions, thousands, etc., in cases like the above, both in speaking and writing; hence, the above would usually be expressed, seventy-two million, five hundred eighty-four thousand, six hundred twenty-three.

22. The four fundamental processes of Arithmetic are **addition, subtraction, multiplication,** and **division.** They are called fundamental processes, because all operations in Arithmetic are based upon them.

ADDITION.

23. **Addition** is the *process* of *finding* the *sum* of *two* or *more* numbers. The sign of addition is +. It is read *plus*, and means *more.* Thus, 5 + 6 is read 5 *plus* 6, and means that 5 and 6 are to be added.

24. The sign of equality is =. It is read *equals* or *is equal to.* Thus, 5 + 6 = 11 may be read 5 plus 6 *equals* 11.

25. *Like numbers* can be added, but *unlike numbers* cannot. Thus, 6 dollars *can* be added to 7 dollars, and the *sum* will be 13 dollars, but 6 dollars *cannot* be added to 7 *feet.*

26. The following table gives the sum of any two numbers from 1 to 12:

TABLE 1.

1 and 1 are 2	2 and 1 are 3	3 and 1 are 4	4 and 1 are 5
1 and 2 are 3	2 and 2 are 4	3 and 2 are 5	4 and 2 are 6
1 and 3 are 4	2 and 3 are 5	3 and 3 are 6	4 and 3 are 7
1 and 4 are 5	2 and 4 are 6	3 and 4 are 7	4 and 4 are 8
1 and 5 are 6	2 and 5 are 7	3 and 5 are 8	4 and 5 are 9
1 and 6 are 7	2 and 6 are 8	3 and 6 are 9	4 and 6 are 10
1 and 7 are 8	2 and 7 are 9	3 and 7 are 10	4 and 7 are 11
1 and 8 are 9	2 and 8 are 10	3 and 8 are 11	4 and 8 are 12
1 and 9 are 10	2 and 9 are 11	3 and 9 are 12	4 and 9 are 13
1 and 10 are 11	2 and 10 are 12	3 and 10 are 13	4 and 10 are 14
1 and 11 are 12	2 and 11 are 13	3 and 11 are 14	4 and 11 are 15
1 and 12 are 13	2 and 12 are 14	3 and 12 are 15	4 and 12 are 16
5 and 1 are 6	6 and 1 are 7	7 and 1 are 8	8 and 1 are 9
5 and 2 are 7	6 and 2 are 8	7 and 2 are 9	8 and 2 are 10
5 and 3 are 8	6 and 3 are 9	7 and 3 are 10	8 and 3 are 11
5 and 4 are 9	6 and 4 are 10	7 and 4 are 11	8 and 4 are 12
5 and 5 are 10	6 and 5 are 11	7 and 5 are 12	8 and 5 are 13
5 and 6 are 11	6 and 6 are 12	7 and 6 are 13	8 and 6 are 14
5 and 7 are 12	6 and 7 are 13	7 and 7 are 14	8 and 7 are 15
5 and 8 are 13	6 and 8 are 14	7 and 8 are 15	8 and 8 are 16
5 and 9 are 14	6 and 9 are 15	7 and 9 are 16	8 and 9 are 17
5 and 10 are 15	6 and 10 are 16	7 and 10 are 17	8 and 10 are 18
5 and 11 are 16	6 and 11 are 17	7 and 11 are 18	8 and 11 are 19
5 and 12 are 17	6 and 12 are 18	7 and 12 are 19	8 and 12 are 20
9 and 1 are 10	10 and 1 are 11	11 and 1 are 12	12 and 1 are 13
9 and 2 are 11	10 and 2 are 12	11 and 2 are 13	12 and 2 are 14
9 and 3 are 12	10 and 3 are 13	11 and 3 are 14	12 and 3 are 15
9 and 4 are 13	10 and 4 are 14	11 and 4 are 15	12 and 4 are 16
9 and 5 are 14	10 and 5 are 15	11 and 5 are 16	12 and 5 are 17
9 and 6 are 15	10 and 6 are 16	11 and 6 are 17	12 and 6 are 18
9 and 7 are 16	10 and 7 are 17	11 and 7 are 18	12 and 7 are 19
9 and 8 are 17	10 and 8 are 18	11 and 8 are 19	12 and 8 are 20
9 and 9 are 18	10 and 9 are 19	11 and 9 are 20	12 and 9 are 21
9 and 10 are 19	10 and 10 are 20	11 and 10 are 21	12 and 10 are 22
9 and 11 are 20	10 and 11 are 21	11 and 11 are 22	12 and 11 are 23
9 and 12 are 21	10 and 12 are 22	11 and 12 are 23	12 and 12 are 24

This table should be carefully committed to memory. Since 0 has no value, the sum of any number and 0 is the number itself; thus, 17 and 0 are 17.

27. For *addition*, place the numbers to be added directly under each other, taking care to place *units* under *units*, *tens* under *tens*, *hundreds* under *hundreds*, and so on.

When the numbers are thus written, the *right-hand figure* of *one number* is placed *directly under the right-hand figure* of the ***number above it***, thus bringing the unit figures of all the numbers to be added in the same vertical line. Proceed as in the following examples:

28. **Example.—What is the sum of 181, 222, 21, 2, and 418?**

Solution.—

$$\begin{array}{rr} & 181 \\ & 222 \\ & 21 \\ & 2 \\ & 418 \\ \hline \textit{sum} & 789 \text{ Ans.} \end{array}$$

Explanation.—After placing the numbers in proper order, begin at the bottom of the right-hand or *units* column, and add, mentally repeating the different sums. Thus, three and two are five and one are six and two are eight and one are nine, the sum of the numbers in *units* column. Place the 9 directly beneath as the first or *units* figure in the sum.

The sum of the numbers in the next or *tens* column equals 8 *tens*, which is the second or *tens* figure in the sum.

The sum of the numbers in the next or *hundreds* column equals 7 *hundreds*, which is the third or *hundreds* figure in the sum.

The sum or answer is 789.

29. **Example.—What is the sum of 425, 36, 9,215, 4, and 907?**

Solution.—

$$\begin{array}{rr} & 425 \\ & 36 \\ & 9215 \\ & 4 \\ & 907 \\ \hline & 27 \\ & 60 \\ & 1500 \\ & 9000 \\ \hline \textit{sum} & 10587 \text{ Ans.} \end{array}$$

Explanation.—The sum of the numbers in the first or units column is seven and four are eleven and five are sixteen and six are twenty-two and five are twenty-seven, or 27 units; i. e., two tens and seven units. Write 27 as shown.

The sum of the numbers in the second or tens column is six tens, or 60. Write 60 underneath 27 as shown. The sum of the numbers in the third or hundreds column is 15 hundreds, or 1,500. Write 1,500 under the two preceding results as shown. There is only one number in the fourth or thousands column, nine, which represents 9,000. Write 9,000 under the three preceding results. Adding these four results, the sum is 10,587, which is the sum of 425, 36, 9,215, 4, and 907.

Note.—It frequently happens, when adding a long column of figures, that the sum of two numbers, one of which does not occur in the addition table, is required. Thus, in the first column above, the sum of 16 and 6 was required. We know from the table that $6 + 6 = 12$; hence, the first figure of the sum is 2. Now, the sum of any number less than 20 and of any number less than 10 must be less than thirty, since $20 + 10 = 30$; therefore, the sum is 22. Consequently, in cases of this kind, add the first figure of the larger number to the smaller number and, if the result is greater than 9, increase the second figure of the larger number by 1. Thus, $44 + 7 = ?$ $4 + 7 = 11$; hence, $44 + 7 = 51$.

30. The addition may also be performed as follows:

$$\begin{array}{rrl} & 425 & \\ & 36 & \\ & 9215 & \\ & 4 & \\ & 907 & \\ \hline \textit{sum} & 10587 & \text{Ans.} \end{array}$$

Explanation.—The sum of the numbers in *units* column $= 27$ *units*, or 2 *tens* and 7 *units*. Write the 7 *units* as the first or right-hand figure in the sum. Reserve the two *tens* and add them to the figures in *tens* column. The sum of the figures in the *tens* column, plus the 2 *tens* reserved and carried from the *units* column $= 8$, which is written down as the second figure in the sum. There is nothing to carry to the next column, because 8 is less than 10. The sum of the numbers in the next column is 15 *hundreds*, or 1 *thousand* and 5 *hundreds*. Write down the 5 as the third or *hundreds* figure in the sum and carry the 1 to the next column. $1 + 9 = 10$, which is written down at the left of the other figures.

The second method saves space and figures, but the first is to be preferred when adding a long column.

31. Example.—Add the numbers in the column below:

Solution.—

$$\begin{array}{rr} & 890\\ & 82\\ & 90\\ & 393\\ & 281\\ & 80\\ & 770\\ & 83\\ & 492\\ & 80\\ & 383\\ & 84\\ & 191\\ \cline{2-2} \textit{sum} & 3899 \end{array}$$ Ans.

Explanation.—The sum of the digits in the first column equals 19 *units*, or 1 *ten* and 9 *units*. Write down the 9 and carry 1 to the next column. The sum of the digits in the second column + 1 = 109 *tens*, or 10 *hundreds* and 9 *tens*. Write down the 9 and carry the 10 to the next column. The sum of the digits in this column plus the 10 reserved = 38.

The entire sum is 3,899.

32. Rule.—I. *Begin at the right, add each column separately, and write the sum, if it be only one figure, under the column added.*

II. *If the sum of any column consists of two or more figures, put the right-hand figure of the sum under that column, and add the remaining figure or figures to the next column.*

33. Proof.—*To prove addition, add each column from top to bottom. If you obtain the same result as by adding from bottom to top, the work is probably correct.*

EXAMPLES FOR PRACTICE.

34. Find the sum of

	Ans.
(*a*) 104 + 203 + 613 + 214.	(*a*) 1,134.
(*b*) 1,875 + 3,143 + 5,826 + 10,832.	(*b*) 21,676.
(*c*) 4,865 + 2,145 + 8,173 + 40,084.	(*c*) 55,267.
(*d*) 14,204 + 8,173 + 1,065 + 10,042.	(*d*) 33,484.

		Ans.
(*e*)	10,832 + 4,145 + 3,133 + 5,872.	(*e*) 23,982.
(*f*)	214 + 1,231 + 141 + 5,000.	(*f*) 6,586.
(*g*)	123 + 104 + 425 + 126 + 327.	(*g*) 1,105.
(*h*)	6,354 + 2,145 + 2,042 + 1,111 + 3,333.	(*h*) 14,985.

SUBTRACTION.

35. In Arithmetic, **subtraction** is the process of finding how much greater one number is than another.

The greater of the two numbers is called the **minuend.**

The smaller of the two numbers is called the **subtrahend.**

The number left after subtracting the *subtrahend* from the *minuend* is called the **difference,** or **remainder.**

36. The sign of subtraction is – . It is read **minus,** and means *less.* Thus, 12 – 7 is read 12 *minus* 7, and means that 7 is to be taken from 12.

37. Example.—From 7,568 take 3,425.

$$\begin{array}{rr} \text{Solution.—}\quad \textit{minuend} & 7568 \\ \textit{subtrahend} & 3425 \\ \hline \textit{remainder} & 4143 \end{array} \quad \text{Ans.}$$

Explanation.—Begin at the right-hand or *units* column and subtract in succession each figure in the subtrahend from the one directly above it in the minuend, and write the remainders below the line. The result is the entire remainder.

38. When there are more figures in the *minuend* than in the *subtrahend,* and when some figures in the *minuend* are *less* than the figures directly under them in the *subtrahend,* proceed as in the following example:

Example.—From 8,453 take 844.

$$\begin{array}{rr} \text{Solution.—}\quad \textit{minuend} & 8453 \\ \textit{subtrahend} & 844 \\ \hline \textit{remainder} & 7609 \end{array} \quad \text{Ans.}$$

Explanation.—Begin to subtract at the right-hand or *units* column. We cannot take 4 from 3, and must, therefore, borrow 1 from 5 in *tens* column and annex it to the 3 in *units* column. The 1 *ten* = 10 *units*, which added to the 3 in *units* column = 13 *units*. 4 from 13 = 9, the first or *units* figure in the remainder.

Since we borrowed 1 from the 5, only 4 remains; 4 from 4 = 0, the second or *tens* figure. We cannot take 8 from 4, and must, therefore, borrow 1 from 8 in *thousands* column. Since 1 *thousand* = 10 *hundreds*, 10 *hundreds* + 4 *hundreds* = 14 *hundreds*, and 8 from 14 = 6, the third or *hundreds* figure in the remainder.

Since we borrowed 1 from 8, only 7 remains, from which there is nothing to subtract; therefore, 7 is the next figure in the remainder or answer.

The operation of borrowing is placing 1 before the figure following the one from which it is borrowed. In the above example the 1 borrowed from 5 is placed before 3, making it 13, from which we subtract 4. The 1 borrowed from 8 is placed before 4, making 14, from which 8 is taken.

39. Example.—Find the difference between 10,000 and 8,763.

Solution.—

minuend	10000
subtrahend	8763
remainder	1237 Ans.

Explanation.—In the above example we borrow 1 from the second column and place it before 0, making 10; 3 from 10 = 7. In the same way we borrow 1 and place it before the next cipher, making 10; but as we have borrowed 1 from this column and taken it to the *units* column, only 9 remains, from which to subtract 6; 6 from 9 = 3. For the same reason we subtract 7 from 9 and 8 from 9 for the next two figures, and obtain a total remainder of 1,237.

40. Rule.—*Place the subtrahend or smaller number under the minuend or larger number, in the same manner as for addition, and proceed as in Arts.* ***37, 38,*** *and* ***39.***

41. Proof.—*To prove an example in subtraction, add the remainder to the subtrahend. The sum should equal the minuend. If it does not, a mistake has been made, and the work should be done over.*

Proof of the above example:

subtrahend 8763
remainder 1287
minuend 10000

EXAMPLES FOR PRACTICE.

42. From

(*a*) 94,278 take 62,574.	Ans.	(*a*) 31,704.
(*b*) 53,714 take 25,824.		(*b*) 27,890.
(*c*) 71,832 take 58,109.		(*c*) 13,723.
(*d*) 20,804 take 10,408.		(*d*) 10,396.
(*e*) 310,465 take 102,141.		(*e*) 208,324.
(*f*) (81,043 + 1,041) take 14,831.		(*f*) 67,253.
(*g*) (20,482 + 18,216) take 21,214.		(*g*) 17,484.
(*h*) (2,040 + 1,213 + 542) take 3,791.		(*h*) 4.

MULTIPLICATION.

43. To **multiply** a number is to *add* it to itself a certain number of times.

44. **Multiplication** is the process of multiplying one number by another.

The *number* thus added to itself, or the number to be multiplied, is called the **multiplicand.**

The *number* which shows how many times the *multiplicand* is to be taken, or the *number* by which we *multiply*, is called the **multiplier.**

The result obtained by multiplying is called the **product.**

45. The sign of multiplication is ×. It is read *times* or *multiplied by.* Thus, 9 × 6 is read 9 *times* 6, or 9 *multiplied by* 6.

46. It matters not in what order the numbers to be multiplied together are placed. Thus, 6 × 9 is the same as 9 × 6.

47. In the following table, the product of any two numbers (neither of which exceeds twelve) may be found:

TABLE 2.

1 times 1 is 1	2 times 1 are 2	3 times 1 are 3
1 times 2 are 2	2 times 2 are 4	3 times 2 are 6
1 times 3 are 3	2 times 3 are 6	3 times 3 are 9
1 times 4 are 4	2 times 4 are 8	3 times 4 are 12
1 times 5 are 5	2 times 5 are 10	3 times 5 are 15
1 times 6 are 6	2 times 6 are 12	3 times 6 are 18
1 times 7 are 7	2 times 7 are 14	3 times 7 are 21
1 times 8 are 8	2 times 8 are 16	3 times 8 are 24
1 times 9 are 9	2 times 9 are 18	3 times 9 are 27
1 times 10 are 10	2 times 10 are 20	3 times 10 are 30
1 times 11 are 11	2 times 11 are 22	3 times 11 are 33
1 times 12 are 12	2 times 12 are 24	3 times 12 are 36
4 times 1 are 4	5 times 1 are 5	6 times 1 are 6
4 times 2 are 8	5 times 2 are 10	6 times 2 are 12
4 times 3 are 12	5 times 3 are 15	6 times 3 are 18
4 times 4 are 16	5 times 4 are 20	6 times 4 are 24
4 times 5 are 20	5 times 5 are 25	6 times 5 are 30
4 times 6 are 24	5 times 6 are 30	6 times 6 are 36
4 times 7 are 28	5 times 7 are 35	6 times 7 are 42
4 times 8 are 32	5 times 8 are 40	6 times 8 are 48
4 times 9 are 36	5 times 9 are 45	6 times 9 are 54
4 times 10 are 40	5 times 10 are 50	6 times 10 are 60
4 times 11 are 44	5 times 11 are 55	6 times 11 are 66
4 times 12 are 48	5 times 12 are 60	6 times 12 are 72
7 times 1 are 7	8 times 1 are 8	9 times 1 are 9
7 times 2 are 14	8 times 2 are 16	9 times 2 are 18
7 times 3 are 21	8 times 3 are 24	9 times 3 are 27
7 times 4 are 28	8 times 4 are 32	9 times 4 are 36
7 times 5 are 35	8 times 5 are 40	9 times 5 are 45
7 times 6 are 42	8 times 6 are 48	9 times 6 are 54
7 times 7 are 49	8 times 7 are 56	9 times 7 are 63
7 times 8 are 56	8 times 8 are 64	9 times 8 are 72
7 times 9 are 63	8 times 9 are 72	9 times 9 are 81
7 times 10 are 70	8 times 10 are 80	9 times 10 are 90
7 times 11 are 77	8 times 11 are 88	9 times 11 are 99
7 times 12 are 84	8 times 12 are 96	9 times 12 are 108
10 times 1 are 10	11 times 1 are 11	12 times 1 are 12
10 times 2 are 20	11 times 2 are 22	12 times 2 are 24
10 times 3 are 30	11 times 3 are 33	12 times 3 are 36
10 times 4 are 40	11 times 4 are 44	12 times 4 are 48
10 times 5 are 50	11 times 5 are 55	12 times 5 are 60
10 times 6 are 60	11 times 6 are 66	12 times 6 are 72
10 times 7 are 70	11 times 7 are 77	12 times 7 are 84
10 times 8 are 80	11 times 8 are 88	12 times 8 are 96
10 times 9 are 90	11 times 9 are 99	12 times 9 are 108
10 times 10 are 100	11 times 10 are 110	12 times 10 are 120
10 times 11 are 110	11 times 11 are 121	12 times 11 are 132
10 times 12 are 120	11 times 12 are 132	12 times 12 are 144

This table should be carefully committed to memory.

Since 0 has no value. the product of 0 and any number is 0

48. To multiply a number by one figure only:

EXAMPLE.—Multiply 425 by 5.

SOLUTION.— *multiplicand* 425
multiplier 5
product 2125 Ans.

EXPLANATION.—For convenience, the *multiplier* is generally written *under* the *right-hand figure* of the *multiplicand*. On looking in the multiplication table, we see that 5 × 5 are 25. *Multiplying* the *first figure* at the *right* of the *multiplicand*, or 5, by the *multiplier* 5, it is seen that 5 times 5 units are 25 units, or 2 tens and 5 units. Write the *5 units* in *units place* in the *product*, and *reserve* the 2 tens to *add* to the *product* of *tens*. Looking in the multiplication table again, we see that 5 × 2 are 10. *Multiplying* the *second figure* of the *multiplicand* by the *multiplier* 5, we see *that* 5 *times* 2 tens are 10 tens, *plus* the 2 tens *reserved*, are 12 tens, or 1 hundred plus 2 tens. Write the *2 tens* in tens place, and *reserve* the 1 hundred to *add* to the product of *hundreds*. Again, we see by the multiplication table that 5 × 4 are 20. *Multiplying* the *third* or *last figure* of the *multiplicand* by the *multiplier* 5, we see that 5 times 4 hundreds are 20 hundreds, *plus* the 1 hundred *reserved*, are 21 hundreds, or 2 thousands *plus* 1 hundred, which we write in *thousands* and *hundreds places*, respectively.

Hence, the *product* is 2,125.

This *result* is the same as adding 425 five times. Thus,

425
425
425
425
425
sum 2125 Ans.

EXAMPLES FOR PRACTICE.

49. Find the product of

(*a*)	61,483 × 6.	Ans.	(*a*)	368,898.
(*b*)	12,375 × 5.		(*b*)	61,875.
(*c*)	10,426 × 7.		(*c*)	72,982.
(*d*)	10,835 × 3.		(*d*)	32.505.

(*e*) 98,376 × 4.	Ans. (*e*) 393,504.
(*f*) 10,873 × 8.	(*f*) 86,984.
(*g*) 71,543 × 9.	(*g*) 643,887.
(*h*) 218,734 × 2.	(*h*) 437,468.

50. To multiply a number by two or more figures:

Example.—Multiply 475 by 234.

Solution.—

$$\begin{array}{rr} \textit{multiplicand} & 475 \\ \textit{multiplier} & 234 \\ \cline{2-2} & 1900 \\ & 1425 \\ & 950 \\ \cline{2-2} \textit{product} & 111150 \end{array} \text{ Ans.}$$

Explanation.—For convenience, the *multiplier* is generally written *under* the *multiplicand*, placing units under units, tens under tens, etc.

We *cannot* multiply by 234 at one operation; we must, therefore, *multiply* by the *parts* and then *add* the **partial products.**

The parts by which we are to multiply are 4 units, 3 tens, and 2 hundreds. 4 times 475 = 1,900, the *first partial product;* 3 times 475 = 1,425, the *second partial product*, the *right-hand figure* of which is *written directly under* the *figure multiplied by*, or 3; 2 times 475 = 950, the *third partial product*, the *right-hand figure* of which is *written directly under* the *figure multiplied by*, or 2.

The sum of these *three partial products* is 111,150, which is the *entire product.*

51. Rule.—I. *Write the multiplier under the multiplicand, so that units are under units, tens under tens, etc.*

II. *Begin at the right and multiply each figure of the multiplicand by each successive figure of the multiplier, placing the right-hand figure of each partial product directly under the figure used as a multiplier.*

III. *The sum of the partial products will equal the required product.*

52. Proof.—*Review the work carefully, or multiply the multiplier by the multiplicand; if the results agree, the work is correct.*

53. When there is a *cipher* in the *multiplier*, multiply the entire multiplicand by it; since the result will be zero, place a cipher under the cipher in the multiplier. Thus,

$$\begin{array}{r} (a) \\ 0 \\ \times 0 \\ \hline 0 \end{array}\text{ Ans.}\qquad \begin{array}{r} (b) \\ 2 \\ \times 0 \\ \hline 0 \end{array}\text{ Ans.}\qquad \begin{array}{r} (c) \\ 15 \\ \times\ 0 \\ \hline 0 \end{array}\text{ Ans.}\qquad \begin{array}{r} (d) \\ 708 \\ \times\ \ 0 \\ \hline 0 \end{array}\text{ Ans.}$$

$$\begin{array}{r} (e) \\ 3114 \\ 203 \\ \hline 9342 \\ 62280 \\ \hline 632142 \end{array}\text{ Ans.}\qquad \begin{array}{r} (f) \\ 4008 \\ 305 \\ \hline 20040 \\ 120240 \\ \hline 1222440 \end{array}\text{ Ans.}\qquad \begin{array}{r} (g) \\ 31264 \\ 1002 \\ \hline 62528 \\ 3126400 \\ \hline 31326528 \end{array}\text{ Ans.}$$

In examples (*e*), (*f*), and (*g*), we multiply by 0 as directed above; then multiply by the next figure of the multiplier and place the first figure of the product alongside the 0, as shown.

EXAMPLES FOR PRACTICE.

54. Find the product of

			Ans.
(*a*)	3,842 × 26.	(*a*)	99,892.
(*b*)	3,716 × 45.	(*b*)	167,220.
(*c*)	1,817 × 124.	(*c*)	225,308.
(*d*)	675 × 38.	(*d*)	25,650.
(*e*)	1,875 × 33.	(*e*)	61,875.
(*f*)	4,836 × 47.	(*f*)	227,292.
(*g*)	5,682 × 543.	(*g*)	3,085,326
(*h*)	3,257 × 246.	(*h*)	801,222.
(*i*)	2,875 × 302.	(*i*)	868,250.
(*j*)	17,819 × 1,004.	(*j*)	17,890,276.
(*k*)	38,674 × 205.	(*k*)	7,928,170.
(*l*)	18,304 × 100.	(*l*)	1,830,400.
(*m*)	7,834 × 10.	(*m*)	78,340.
(*n*)	87,543 × 1,000.	(*n*)	87,543,000.
(*o*)	48,763 × 100.	(*o*)	4,876,300.

DIVISION.

55. Division is the process of finding how many times one number is contained in another of the same kind.

The number to be *divided* is called the **dividend.**

The number by which we *divide* is called the **divisor.**

The number which *shows* how many times the *divisor* is *contained* in the *dividend* is called the **quotient.**

56. The sign of division is ÷. It is read *divided by.* $54 \div 9$ is read 54 *divided by* 9. Another way to write 54 *divided by* 9 is $\frac{54}{9}$. Thus, $54 \div 9 = 6$, or $\frac{54}{9} = 6$.

In both of these cases 54 is the *dividend* and 9 is the *divisor.*

Division is the *reverse* of **multiplication.**

57. To divide when the divisor consists of but one figure, proceed as in the following example:

EXAMPLE.—What is the quotient of 875 ÷ 7?

SOLUTION.—

$$\begin{array}{rcccccl}
 & \textit{divisor} & \multicolumn{3}{c}{\textit{dividend}} & \textit{quotient} \\
 & 7) & 8 & 7 & 5 & (125 & \text{Ans.} \\
 & & 7 & & & & \\
\cline{3-3}
 & & 1 & 7 & & & \\
 & & 1 & 4 & & & \\
\cline{3-4}
 & & & 3 & 5 & & \\
 & & & 3 & 5 & & \\
\cline{4-5}
\textit{remainder} & & & & 0 & &
\end{array}$$

EXPLANATION.—7 is contained in 8 *hundreds* 1 *hundred* times. Place the one as the first or left-hand figure of the quotient. Multiply the divisor 7 by the 1 *hundred* of the quotient, and place the product 7 *hundreds* under the 8 *hundreds* in the dividend, and subtract. Beside the remainder 1, bring down the next or *tens* figure of the dividend, in this case 7, making 17 *tens;* 7 is contained in 17, 2 times. Write the 2 as the second figure of the quotient. Multiply the divisor 7 by the 2 in the quotient, and subtract the product from 17. Beside the remainder 3, bring down the next or *units* figure of the dividend, in this case 5, making

35 *units.* 7 is contained in 35, 5 times, which is placed in the quotient. Multiplying the divisor by the last figure of the quotient, 5 times 7 = 35, which subtracted from 35, under which it is placed, leaves 0. Therefore, the quotient is 125. This method is called **long division.**

58. In **short division,** only the divisor, dividend, and quotient are written, the operations being performed mentally.

$$\begin{array}{rl} & \textit{dividend} \\ \textit{divisor} & 7\,)\,8^{1}7^{3}5 \\ \hline \textit{quotient} & \;\;1\;\;2\;\;5 \quad \text{Ans.} \end{array}$$

The mental operation is as follows: 7 is contained in 8, once and one remainder; 1 placed before 7 makes 17; 7 is contained in 17, 2 times and 3 over; the 3 placed before 5 makes 35; 7 is contained in 35, 5 times. These partial quotients placed in order as they are found, make the entire quotient 125.

The small figures are placed in the example given to better illustrate the explanation; they are never written when actually performing division in this way.

59. If the *divisor* consists of *2 or more* figures, proceed as in the following example:

EXAMPLE.—Divide 2,702,826 by 63.

SOLUTION.—

$$\begin{array}{rcccccccl}
\textit{divisor} & & & & \textit{dividend} & & & & \textit{quotient} \\
63\,) & 2 & 7 & 0 & 2 & 8 & 2 & 6 & (\,42902 \ \text{Ans.} \\
 & 2 & 5 & 2 & & & & & \\
\cline{2-4}
 & & 1 & 8 & 2 & & & & \\
 & & 1 & 2 & 6 & & & & \\
\cline{2-5}
 & & & 5 & 6 & 8 & & & \\
 & & & 5 & 6 & 7 & & & \\
\cline{4-8}
 & & & & & 1 & 2 & 6 & \\
 & & & & & 1 & 2 & 6 & \\
\cline{7-8}
 & & & & & & & 0 &
\end{array}$$

EXPLANATION.—As 63 is not contained in the first two figures, 27, we must use the first three figures, 270. Now, by trial, we must find how many times 63 is contained in 270;

6 is contained in the first two figures of 270, 4 times. Place the 4 as the first or left-hand figure in the quotient. Multiply the divisor 63 by 4, and subtract the product 252 from 270. The remainder is 18, beside which we write the next figure of the dividend, 2, making 182. Now, 6 is contained in the first two figures of 182, 3 times, but on multiplying 63 by 3, we see that the product 189 is too great, so we try 2 as the second figure of the quotient. Multiplying the divisor 63 by 2, and subtracting the product 126 from 182, the remainder is 56, beside which we bring down the next figure of the dividend, making 568; 6 is contained in 56 about 9 times. Multiply the divisor 63 by 9 and subtract the product 567 from 568. The remainder is 1, and bringing down the next figure of the dividend, 2, gives 12. As 12 is smaller than 63, we write 0 in the quotient and bring down the next figure, 6, making 126. 63 is contained in 126, 2 times, without a remainder. Therefore, 42,902 is the quotient.

60. Rule.—I. *Write the divisor at the left of the dividend, with a line between them.*

II. *Find how many times the divisor is contained in the lowest number of the left-hand figures of the dividend that will contain it, and write the result at the right of the dividend, with a line between, for the first figure of the quotient.*

III. *Multiply the divisor by this quotient; write the product under the partial dividend used, and subtract, annexing to the remainder the next figure of the dividend. Divide as before, and thus continue until all the figures of the dividend have been used.*

IV. *If any partial dividend will not contain the divisor, write a cipher in the quotient, annex the next figure of the dividend and proceed as before.*

V. *If there be a remainder at last, write it after the quotient, with the divisor underneath.*

61. Proof.—*Multiply the quotient by the divisor, and add the remainder, if there be any, to the product. The result will be the dividend.*

divisor dividend quotient
Thus, 63)4235(67$\frac{14}{63}$ Ans.
378
455
441
remainder 14
Proof, *quotient* 67
divisor 63
201
402
4221
remainder 14
dividend 4235

EXAMPLES FOR PRACTICE.

62. Divide the following:

(*a*) 126,498 by 58.	Ans. (*a*) 2,181.
(*b*) 3,207,594 by 767.	(*b*) 4,182.
(*c*) 11,408,202 by 234.	(*c*) 48,753.
(*d*) 2,100,315 by 581.	(*d*) 3,615.
(*e*) 969,936 by 4,008.	(*e*) 242.
(*f*) 7,481,888 by 1,021.	(*f*) 7,328.
(*g*) 1,525,915 by 5,003.	(*g*) 305.
(*h*) 1,646,301 by 381.	(*h*) 4,321.

CANCELATION.

63. Cancelation is the process of shortening operations in division by casting out equal factors from both dividend and divisor.

64. The **factors** of a number are *those numbers* which, when *multiplied* together, will *equal that number.* Thus, 5 and 3 are factors of 15, since 5 × 3 = 15. Likewise, 8 and 7 are the factors of 56, since 8 × 7 = 56.

65. A **prime number** is one which cannot be divided by any number except itself and 1. Thus, 2, 3, 11, 29, etc., are prime numbers.

66. A **prime factor** is any factor that is a prime number.

Any number that is not a prime is called a **composite** number, and may be produced by multiplying together its prime factors. Thus, 60 is a composite number, and is equal to the product of its prime factors, $2 \times 2 \times 3 \times 5$.

Numbers are said to be **prime to each other** when no two of them can be divided by any number except 1; the numbers themselves *may* be either prime or composite. Thus, the numbers 3, 5, and 11 are prime to each other, so also are 22, 25, and 21 — all composite numbers.

67. Canceling *equal factors* from *both dividend and divisor* does *not* change the *quotient.*

The *canceling* of a *factor* in *both dividend and divisor* is the *same* as *dividing them both* by the *same number*, which, by the principle of division, does not *change the quotient.*

Write the *numbers* which make the *dividend* above the *line*, and those which make the *divisor* below it.

68. EXAMPLE.—Divide $4 \times 45 \times 60$ by 9×24.

SOLUTION.—Placing the dividend over the divisor, and canceling

$$\frac{\cancel{4} \times \overset{5}{\cancel{45}} \times \overset{10}{\cancel{60}}}{\cancel{9} \times \underset{\underset{1}{\cancel{6}}}{\cancel{24}}} = \frac{50}{1} = 50. \quad \text{Ans.}$$

EXPLANATION.—The 4 in the dividend and 24 in the divisor are both divisible by 4, since 4 divided by 4 equals 1, and 24 divided by 4 equals 6. Cross off the four and write the 1 over it; also, cross off the 24 and write the 6 under it. Thus,

$$\frac{\overset{1}{\cancel{4}} \times 45 \times 60}{9 \times \underset{6}{\cancel{24}}}.$$

60 in the dividend and 6 in the divisor are divisible by 6, since 60 divided by 6 equals 10, and 6 divided by 6 equals 1. Cross off the 60 and write 10 over it; also, cross off the 6 and write 1 under it. Thus,

$$\frac{\overset{1}{\cancel{4}} \times 45 \times \overset{10}{\cancel{60}}}{9 \times \underset{\underset{1}{\cancel{6}}}{\cancel{24}}}.$$

Again, 45 in the dividend and 9 in the divisor are divisible by 9, since 45 divided by 9 equals 5, and 9 divided by 9 equals 1. Cross off the 45 and write the 5 over it; also, cross off the 9 and write the 1 under it. Thus,

$$\frac{\overset{1}{\cancel{4}}\times\overset{5}{\cancel{45}}\times\overset{10}{\cancel{60}}}{\underset{1}{\cancel{9}}\times\underset{\underset{1}{\cancel{6}}}{\cancel{24}}}.$$

Since there are no two remaining numbers (one in the dividend and one in the divisor) divisible by any number except 1, without a remainder, it is impossible to cancel further.

Multiply all the uncanceled numbers in the dividend together, and divide their product by the product of all the uncanceled numbers in the divisor. The result will be the quotient. The product of all the uncanceled numbers in the dividend equals $5 \times 1 \times 10 = 50$; the product of all the uncanceled numbers in the divisor equals $1 \times 1 = 1$.

$$\text{Hence, } \frac{\overset{1}{\cancel{4}}\times\overset{5}{\cancel{45}}\times\overset{10}{\cancel{60}}}{\underset{1}{\cancel{9}}\times\underset{\underset{1}{\cancel{6}}}{\cancel{24}}}=\frac{1\times5\times10}{1\times1}=50. \quad \text{Ans.}$$

It is usual to omit the 1's when canceling them, instead of writing them as above.

69. Rule.—I. *Cancel the common factors from both the dividend and divisor.*

II. *Then divide the product of the remaining factors of the dividend by the product of the remaining factors of the divisor, and the result will be the quotient.*

EXAMPLES FOR PRACTICE.

70. Divide

	Ans.
(*a*) $14 \times 18 \times 16 \times 40$ by $7 \times 8 \times 6 \times 5 \times 3$.	(*a*) 32.
(*b*) $3 \times 65 \times 50 \times 100 \times 60$ by $30 \times 60 \times 13 \times 10$.	(*b*) 250.
(*c*) $8 \times 4 \times 3 \times 9 \times 11$ by $11 \times 9 \times 4 \times 3 \times 8$.	(*c*) 1.
(*d*) $164 \times 321 \times 6 \times 7 \times 4$ by $82 \times 321 \times 7$.	(*d*) 48.
(*e*) $50 \times 100 \times 200 \times 72$ by $1,000 \times 144 \times 100$.	(*e*) 5.
(*f*) $48 \times 63 \times 55 \times 49$ by $7 \times 21 \times 11 \times 48$.	(*f*) 105.
(*g*) $110 \times 150 \times 84 \times 32$ by $11 \times 15 \times 100 \times 64$.	(*g*) 42.
(*h*) $115 \times 120 \times 400 \times 1,000$ by $23 \times 1,000 \times 60 \times 800$.	(*h*) 5.

ARITHMETIC.

(SECTION 2.)

FRACTIONS.

71. A **fraction** is one or more equal parts of a unit: *One-half*, *one-third*, *two-fifths* are fractions.

72. *Two* numbers are required to express a fraction; one is called the *numerator*, and the other the *denominator*.

73. The numerator is placed above the denominator, with a line between them; as, $\frac{2}{3}$. 3 is the **denominator,** and shows into how many *equal parts* the *unit* or *one* is divided. The **numerator** 2 shows how many of these equal parts are taken or considered. The denominator also indicates the names of the parts.

$\frac{1}{2}$ is read one-half; $\frac{3}{4}$ is read three-fourths; $\frac{3}{8}$ is read three-eighths; $\frac{5}{16}$ is read five-sixteenths; $\frac{29}{47}$ is read twenty-nine-forty-sevenths.

74. In the expression "$\frac{3}{4}$ of an apple," the denominator 4 shows that the apple is to be (or has been) cut into 4 *equal parts*, and the numerator 3 shows that *three of these parts*, or *fourths*, are taken or considered.

If each of the parts, or fourths, of the apple were cut in *two equal pieces*, there would then be twice as many pieces as before, or $4 \times 2 = 8$ pieces in all; one of these pieces would be called one-eighth, and would be expressed in figures as $\frac{1}{8}$. Three of these pieces would be called three-eighths, and written $\frac{3}{8}$. It is evident that the *larger the denominator*, the greater is the number of parts into which anything is divided and the less the value of the fraction for the same number of parts taken. In other words, $\frac{7}{9}$, for example, is smaller than $\frac{7}{8}$, because if an object be divided into 9 parts, the parts are smaller than if the same object had been divided into 8 parts; and, since $\frac{1}{9}$ is smaller than $\frac{1}{8}$,

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it is clear that 7 one-ninths is a smaller amount than 7 one-eighths. Hence, also, $\frac{7}{9}$ is less than $\frac{7}{8}$.

75. The **value** of a fraction is the *numerator* divided by the *denominator;* as, $\frac{4}{2} = 2$, $\frac{6}{2} = 3$.

76. The line between the *numerator* and *denominator* means *divided by*, or $\div$.

$\frac{3}{4}$ is equivalent to $3 \div 4$.
$\frac{5}{8}$ is equivalent to $5 \div 8$.

77. The *numerator* and *denominator* of a fraction, when considered together, are called the **terms** of a fraction.

78. The *value* of a fraction whose *numerator* and *denominator* are equal is 1.

$\frac{4}{4}$, or four-fourths, $= 1$.
$\frac{8}{8}$, or eight-eighths, $= 1$.
$\frac{64}{64}$, or sixty-four-sixty-fourths, $= 1$.

79. A **proper fraction** is a fraction whose *numerator* is *less* than its *denominator*. Its *value* is *less* than 1; as, $\frac{3}{4}$, $\frac{5}{8}$, $\frac{1}{16}$.

80. An **improper fraction** is a fraction whose *numerator equals* or is *greater than* the *denominator*. Its *value* is *one* or *more* than *one;* as, $\frac{4}{4}$, $\frac{9}{8}$, $\frac{42}{32}$.

81. A **mixed number** is a *whole* number and a *fraction* united. $4\frac{2}{3}$ is a mixed number, and is equivalent to $4 + \frac{2}{3}$. It is read *four and two-thirds*.

REDUCTION OF FRACTIONS.

82. **Reduction of fractions** is the process of changing their form without changing their *value*.

83. A *fraction* is reduced to *higher terms* by *multiplying both terms* of the *fraction* by the *same number*. Thus, $\frac{3}{4}$ is reduced to $\frac{6}{8}$ by multiplying both terms by 2.

$$\frac{3 \times 2}{4 \times 2} = \frac{6}{8}.$$

The *value* is not changed, since $\frac{3}{4} = \frac{6}{8}$. For, suppose that an object, say an apple, is divided into 8 equal parts. If these parts be arranged into 4 piles, each containing 2 parts, it is evident that each pile will be composed of the same amount of the entire apple as would have been the case had the apple been originally cut into 4 equal parts. Now, if one of these piles (containing 2 parts) be removed, there will be 3 piles left, each containing 2 equal parts, or 6 equal parts in all, i. e., six-eighths. But, since one pile, or one quarter, was removed, there are three-quarters left. Hence, $\frac{3}{4} = \frac{6}{8}$. The same course of reasoning may be applied to any similar case. Therefore, multiplying both terms of a fraction by the same number does not alter its value.

84. To reduce a fraction to an equal fraction having a given denominator:

Example.—Reduce $\frac{7}{8}$ to an equal fraction having 96 for a denominator.

Solution.—Both the numerator and the denominator must be multiplied by the same number in order not to change the value of the fraction. The denominator must be multiplied by some number which will, in this case, make the product 96; this number is evidently $96 \div 8 = 12$, since $8 \times 12 = 96$. Hence, $\frac{7 \times 12}{8 \times 12} = \frac{84}{96}$. Ans.

85. Rule.—*Divide the given denominator by the denominator of the given fraction, and multiply both terms of the fraction by the result.*

Example.—Reduce $\frac{3}{4}$ to 100ths.

Solution.— $100 \div 4 = 25$; hence, $\frac{3 \times 25}{4 \times 25} = \frac{75}{100}$. Ans.

86. A *fraction* is reduced to *lower terms* by *dividing both terms* by the *same number*. Thus, $\frac{8}{10}$ is reduced to $\frac{4}{5}$ by dividing both terms by 2.

$$\frac{8 \div 2}{10 \div 2} = \frac{4}{5}.$$

That $\frac{8}{10} = \frac{4}{5}$ is readily seen from the explanation given in Art. **83**; for, multiplying both terms of the fraction $\frac{4}{5}$ by 2, $\frac{4 \times 2}{5 \times 2} = \frac{8}{10}$, and, if $\frac{4}{5} = \frac{8}{10}$, $\frac{8}{10}$ must equal $\frac{4}{5}$. Hence, dividing both terms of a fraction by the same number does not alter its value.

87. A *fraction* is reduced to *lowest terms* when its *numerator and denominator* cannot both be *divided* by the *same* *number* without a remainder; as, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{11}{12}$, $\frac{8}{15}$. In other words, the numerator and denominator are prime to each other.

EXAMPLES FOR PRACTICE.

88. Reduce the following:

(*a*) $\frac{7}{16}$ to 128ths.	Ans. (*a*) $\frac{56}{128}$.
(*b*) $\frac{24}{132}$ to its lowest terms.	(*b*) $\frac{2}{11}$.
(*c*) $\frac{64}{1000}$ to its lowest terms.	(*c*) $\frac{8}{125}$.
(*d*) $\frac{5}{7}$ to 49ths.	(*d*) $\frac{35}{49}$.
(*e*) $\frac{13}{16}$ to 10,000ths.	(*e*) $\frac{8125}{10000}$.

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89. To reduce a whole number or mixed number to an improper fraction:

Example.—How many *fourths* in 5?

Solution.—Since there are 4 *fourths* in 1 ($\frac{4}{4} = 1$), in 5 there will be 5×4 fourths, or 20 fourths; i. e., $5 \times \frac{4}{4} = \frac{20}{4}$. Ans.

Example.—Reduce $8\frac{3}{4}$ to an improper fraction.

Solution.—$8 \times \frac{4}{4} = \frac{32}{4}$. $\frac{32}{4} + \frac{3}{4} = \frac{35}{4}$. Ans.

90. Rule.—*Multiply the whole number by the denominator of the fraction, add the numerator to the product, and place the denominator under the result. If it is desired to reduce a whole number to a fraction, multiply the whole number by the denominator of the given fraction, and write the result over the denominator.*

EXAMPLES FOR PRACTICE.

91. Reduce to improper fractions:

(*a*) $4\frac{2}{5}$.	Ans. (*a*) $\frac{22}{5}$.
(*b*) $5\frac{1}{9}$.	(*b*) $\frac{46}{9}$.
(*c*) $10\frac{2}{10}$.	(*c*) $\frac{102}{10}$.
(*d*) $37\frac{3}{4}$.	(*d*) $\frac{151}{4}$.
(*e*) $50\frac{4}{5}$.	(*e*) $\frac{254}{5}$.
(*f*) Reduce 7 to a fraction whose denominator is 16.	(*f*) $\frac{112}{16}$.

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92. To reduce an improper fraction to a whole or mixed number:

Example.—Reduce $\frac{21}{4}$ to a mixed number.

Solution.—4 is contained in 21, 5 times and 1 remaining (see Art. **75**); as this is also divided by 4, its value is $\frac{1}{4}$. Therefore, $5 + \frac{1}{4}$, or $5\frac{1}{4}$, is the number. Ans.

93. Rule.—*Divide the numerator by the denominator, the quotient will be the whole number; the remainder, if there be any, will be the numerator of the fractional part of which the denominator is the same as the denominator of the improper fraction.*

EXAMPLES FOR PRACTICE.

94. Reduce to whole or mixed numbers:

(*a*) $\frac{145}{6}$.	Ans.	(*a*) $24\frac{1}{6}$.
(*b*) $\frac{185}{3}$.		(*b*) $61\frac{2}{3}$.
(*c*) $\frac{701}{6}$.		(*c*) $116\frac{5}{6}$.
(*d*) $\frac{148}{3}$.		(*d*) $49\frac{1}{3}$.
(*e*) $\frac{76}{19}$.		(*e*) 4.
(*f*) $\frac{125}{25}$.		(*f*) 5.

95. A **common denominator** of *two* or *more fractions* is a number which will contain (i. e., which may be divided by) all of the *denominators* of the *fractions* without a remainder. The **least common denominator** is the least number that will contain all of the denominators of the fractions without a remainder.

96. To find the least common denominator:

Example.—Find the least common denominator of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{9}$, and $\frac{1}{16}$.

Solution.—We first place the denominators in a row, separated by commas.

$$\begin{array}{r|rrrr} 2 & 4, & 3, & 9, & 16 \\ \hline 2 & 2, & 3, & 9, & 8 \\ \hline 3 & 1, & 3, & 9, & 4 \\ \hline & 1, & 1, & 3, & 4 \end{array}$$

$2 \times 2 \times 3 \times 3 \times 4 = 144$, the least common denominator. Ans.

Explanation.—Divide the numbers by some prime number that will divide at least two of them without a remainder (if possible), bringing down to the row below those denominators which will not contain the divisor without a remainder. Dividing each of the numbers by 2, the second row becomes 2, 3, 9, 8, since 2 will not divide 3 and 9 without a remainder. Dividing again by 2, the result is 1, 3, 9, 4.

Dividing the third row by 3, the result is 1, 1, 3, 4. Since the remaining numbers are prime to each other, we cease dividing further. The product of all the divisors and of the numbers prime to each other, is $2 \times 2 \times 3 \times 3 \times 4 = 144$, which is the required least common denominator.

97. Example.—Find the least common denominator of $\frac{4}{9}$, $\frac{5}{12}$, $\frac{7}{18}$.

Solution.—

$$\begin{array}{r|rrr} 3 & 9, & 12, & 18 \\ \hline 3 & 3, & 4, & 6 \\ \hline 2 & 1, & 4, & 2 \\ \hline & 1, & 2, & 1 \end{array}$$

$3 \times 3 \times 2 \times 2 = 36$. Ans.

98. To reduce two or more fractions to fractions having a common denominator:

Example.—Reduce $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{1}{2}$ to fractions having a common denominator.

Solution.—The common denominator is any number which will contain 3, 4, and 2. The *least* common denominator is 12, because it is the smallest number which can be divided by 3, 4, and 2 without a remainder. $\frac{2}{3} = \frac{8}{12}$, $\frac{3}{4} = \frac{9}{12}$, $\frac{1}{2} = \frac{6}{12}$.

Reducing $\frac{2}{3}$ (see Art. **84**), 3 is contained in 12, 4 times. By multiplying both numerator and denominator of $\frac{2}{3}$ by 4, we find

$\frac{2 \times 4}{3 \times 4} = \frac{8}{12}$. In the same way we find $\frac{3}{4} = \frac{9}{12}$ and $\frac{1}{2} = \frac{6}{12}$.

99. Rule.—*Divide the common denominator by the denominator of the given fraction, and multiply both terms of the fraction by the quotient.*

EXAMPLES FOR PRACTICE.

100. Reduce to fractions having a common denominator:

(*a*) $\frac{3}{4}$, $\frac{5}{8}$, $\frac{7}{8}$.		(*a*) $\frac{6}{8}$, $\frac{5}{8}$, $\frac{7}{8}$.
(*b*) $\frac{3}{16}$, $\frac{3}{4}$, $\frac{7}{32}$.		(*b*) $\frac{6}{32}$, $\frac{24}{32}$, $\frac{7}{32}$.
(*c*) $\frac{7}{8}$, $\frac{7}{88}$, $\frac{10}{11}$.	Ans.	(*c*) $\frac{77}{88}$, $\frac{7}{88}$, $\frac{80}{88}$.
(*d*) $\frac{3}{5}$, $\frac{5}{8}$, $\frac{11}{40}$.		(*d*) $\frac{24}{40}$, $\frac{25}{40}$, $\frac{11}{40}$.
(*e*) $\frac{4}{10}$, $\frac{6}{40}$, $\frac{9}{20}$.		(*e*) $\frac{16}{40}$, $\frac{6}{40}$, $\frac{18}{40}$.
(*f*) $\frac{7}{15}$, $\frac{17}{30}$, $\frac{21}{30}$.		(*f*) $\frac{14}{30}$, $\frac{17}{30}$, $\frac{21}{30}$.

ADDITION OF FRACTIONS.

101. *Fractions cannot be added unless they have a common denominator.* We cannot add $\frac{3}{4}$ to $\frac{7}{8}$ as they now stand, since the denominators represent parts of different sizes. Fourths cannot be added to eighths.

Suppose we divide an apple into 4 equal parts, and then divide 2 of these parts into two equal parts. It is evident that we shall have 2 one-fourths and 4 one-eighths. Now, if we add these parts, the result is $2 + 4 = 6$ something. But what is this something ? It is not fourths, for six fourths are $1\frac{1}{2}$, and we had only 1 apple to begin with ; neither is it eighths, for six eighths are $\frac{3}{4}$, which is less than 1 apple. By reducing the quarters to eighths, we have $\frac{2}{4} = \frac{4}{8}$, and adding the other 4 eighths, $4 + 4 = 8$ eighths. This result is correct, since $\frac{8}{8} = 1$. Or, we can, in this case, reduce the eighths to quarters. Thus, $\frac{4}{8} = \frac{2}{4}$; whence, adding $2 + 2 = 4$ quarters, a correct result since $\frac{4}{4} = 1$.

Before adding, fractions should be reduced to a common denominator, preferably the *least* common denominator.

102. EXAMPLE.—Find the sum of $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{8}$.

SOLUTION.—The *least common denominator*, or the *least number* which will contain all the *denominators*, is 8.

$$\frac{1}{2} = \frac{4}{8}, \quad \frac{3}{4} = \frac{6}{8}, \quad \frac{5}{8} = \frac{5}{8}.$$

EXPLANATION.—As the *denominator* tells or indicates the names of the *parts*, the *numerators* only are added in order to obtain the total number of *parts* indicated by the *denominator*. Thus, 4 one-eighths plus 6 one-eighths plus 5 one-eighths =

$$\frac{4}{8} + \frac{6}{8} + \frac{5}{8} = \frac{4+6+5}{8} = \frac{15}{8} = 1\frac{7}{8}. \quad \text{Ans.}$$

103. EXAMPLE.—What is the sum of $12\frac{3}{4}$, $14\frac{5}{8}$, and $7\frac{5}{16}$?

SOLUTION.—The least common denominator in this case is 16.

$$\begin{aligned} 12\tfrac{3}{4} &= 12\tfrac{12}{16} \\ 14\tfrac{5}{8} &= 14\tfrac{10}{16} \\ 7\tfrac{5}{16} &= 7\tfrac{5}{16} \\ \hline \textit{sum} &= 33 + \tfrac{27}{16} = 33 + 1\tfrac{11}{16} = 34\tfrac{11}{16}. \quad \text{Ans.} \end{aligned}$$

The sum of the fractions = $\frac{27}{16}$ or $1\frac{11}{16}$, which added to the sum of the whole numbers = $34\frac{11}{16}$.

EXAMPLE.—What is the sum of 17, $13\frac{5}{16}$, $\frac{3}{32}$, and $8\frac{1}{4}$?

SOLUTION.—The least common denominator is 32. $13\frac{5}{16} = 13\frac{10}{32}$, $8\frac{1}{4} = 8\frac{8}{32}$.

$$\begin{array}{r} 17 \\ 13\frac{10}{32} \\ \frac{3}{32} \\ 8\frac{8}{32} \\ \hline \textit{sum}\ 38\frac{21}{32}. \end{array} \quad \text{Ans.}$$

104. Rule I.—*Reduce the given fractions to fractions having the least common denominator, and write the sum of the numerators over the common denominator.*

II. *When there are mixed numbers and whole numbers, add the fractions first, and if their sum is an improper fraction, reduce it to a mixed number, and add the whole number with the other whole numbers.*

EXAMPLES FOR PRACTICE.

105. Find the sum of

(*a*) $\frac{4}{6}$, $\frac{7}{24}$, $\frac{5}{8}$.	Ans.	(*a*) $1\frac{7}{12}$.
(*b*) $\frac{3}{8}$, $\frac{5}{16}$, $\frac{24}{48}$.		(*b*) $1\frac{3}{16}$.
(*c*) $\frac{1}{2}$, $\frac{3}{8}$, $\frac{5}{16}$.		(*c*) $1\frac{3}{16}$.
(*d*) $\frac{5}{6}$, $\frac{11}{72}$, $\frac{13}{18}$.		(*d*) $1\frac{51}{72}$.
(*e*) $\frac{10}{11}$, $\frac{6}{33}$, $\frac{22}{66}$.		(*e*) $1\frac{28}{66}$.
(*f*) $\frac{22}{32}$, $\frac{11}{16}$, $\frac{14}{56}$.		(*f*) $1\frac{5}{8}$.
(*g*) $\frac{4}{11}$, $\frac{7}{33}$, $\frac{14}{22}$.		(*g*) $1\frac{7}{33}$.
(*h*) $\frac{3}{8}$, $\frac{11}{24}$, $\frac{1}{6}$.		(*h*) 1.

SUBTRACTION OF FRACTIONS.

106. Fractions cannot be *subtracted* without first reducing them to a *common denominator.* This can be shown in the same manner as in the case of addition of fractions.

EXAMPLE.—Subtract $\frac{3}{8}$ from $\frac{13}{16}$.

SOLUTION.—The common denominator is 16.

$$\frac{3}{8} = \frac{6}{16}. \quad \frac{13}{16} - \frac{6}{16} = \frac{13-6}{16} = \frac{7}{16}. \quad \text{Ans.}$$

107. EXAMPLE.—From 7 take $\frac{5}{8}$.

SOLUTION.— $1 = \frac{8}{8}$; therefore, since $7 = 6 + 1$, $7 = 6 + \frac{8}{8} = 6\frac{8}{8}$, or $6\frac{8}{8} - \frac{5}{8} = 6\frac{3}{8}$. Ans.

108. Example.—What is the difference between $17\frac{9}{16}$ and $9\frac{15}{32}$?

Solution.—The common denominator of the fractions is 32. $17\frac{9}{16} = 17\frac{18}{32}$.

$$\begin{array}{rr} \textit{minuend} & 17\frac{18}{32} \\ \textit{subtrahend} & 9\frac{15}{32} \\ \hline \textit{difference} & 8\frac{3}{32} \end{array} \text{ Ans.}$$

109. Example.—From $9\frac{1}{4}$ take $4\frac{7}{16}$.

Solution.—The common denominator of the fractions is 16. $9\frac{1}{4} = 9\frac{4}{16}$.

$$\begin{array}{rrr} \textit{minuend} & 9\frac{4}{16} \text{ or} & 8\frac{20}{16} \\ \textit{subtrahend} & 4\frac{7}{16} & 4\frac{7}{16} \\ \hline \textit{difference} & 4\frac{13}{16} & 4\frac{13}{16} \end{array} \text{ Ans.}$$

Explanation.—As the *fraction* in the *subtrahend* is greater than the fraction in the *minuend*, it *cannot* be subtracted; therefore, *borrow* 1, or $\frac{16}{16}$, from the 9 in the *minuend* and *add* it to the $\frac{4}{16}$; $\frac{4}{16} + \frac{16}{16} = \frac{20}{16}$. $\frac{7}{16}$ from $\frac{20}{16} = \frac{13}{16}$. Since 1 was *borrowed* from 9, 8 *remains;* 4 from 8 = 4; $4 + \frac{13}{16} = 4\frac{13}{16}$.

110. Example.—From 9 take $8\frac{3}{16}$.

Solution.—

$$\begin{array}{rrr} \textit{minuend} & 9 \text{ or} & 8\frac{16}{16} \\ \textit{subtrahend} & 8\frac{3}{16} & 8\frac{3}{16} \\ \hline \textit{difference} & \frac{13}{16} & \frac{13}{16} \end{array} \text{ Ans.}$$

Explanation.—As there is no *fraction* in the *minuend* from which to take the *fraction* in the *subtrahend*, *borrow* 1, or $\frac{16}{16}$, from 9. $\frac{3}{16}$ from $\frac{16}{16} = \frac{13}{16}$. Since 1 was *borrowed* from 9, only 8 is left. 8 from 8 = 0.

111. Rule I.—*Reduce the fractions to fractions having a common denominator. Subtract one numerator from the other and place the remainder over the common denominator.*

II. *When there are mixed numbers, subtract the fractions and whole numbers separately, and place the remainders side by side.*

III. *When the fraction in the subtrahend is greater than the fraction in the minuend, borrow 1 from the whole number in the minuend and add it to the fraction in the minuend, from which subtract the fraction in the subtrahend.*

IV. *When the minuend is a whole number, borrow 1; reduce it to a fraction whose denominator is the same as the denominator of the fraction in the subtrahend, and place it over that fraction for subtraction.*

EXAMPLES FOR PRACTICE.

112. Subtract

(*a*) $\frac{10}{24}$ from $\frac{11}{12}$.		(*a*) $\frac{1}{2}$.
(*b*) $\frac{7}{14}$ from $\frac{17}{28}$.		(*b*) $\frac{3}{28}$.
(*c*) $\frac{4}{30}$ from $\frac{5}{10}$.		(*c*) $\frac{11}{30}$.
(*d*) $\frac{15}{35}$ from $\frac{45}{70}$.	Ans.	(*d*) $\frac{3}{14}$.
(*e*) $\frac{15}{16}$ from $\frac{57}{48}$.		(*e*) $\frac{1}{4}$.
(*f*) $13\frac{1}{4}$ from $30\frac{1}{2}$.		(*f*) $17\frac{1}{4}$.
(*g*) $12\frac{1}{8}$ from 27.		(*g*) $14\frac{7}{8}$.
(*h*) $5\frac{1}{4}$ from 30.		(*h*) $24\frac{3}{4}$.

MULTIPLICATION OF FRACTIONS.

113. In *multiplication* of fractions it is not necessary to *reduce* the *fractions* to fractions having a *common denominator.*

114. Multiplying the *numerator* or *dividing* the *denominator multiplies* the fraction.

EXAMPLE.—Multiply $\frac{3}{4}$ by 4.

SOLUTION.— $\frac{3}{4} \times 4 = \frac{3 \times 4}{4} = \frac{12}{4} = 3$. Ans.

Or $\frac{3}{4} \times 4 = \frac{3}{4 \div 4} = \frac{3}{1} = 3$. Ans.

The word "of" in multiplication of fractions means the same as ×, or times. Thus,

$$\frac{3}{4} \text{ of } 4 = \frac{3}{4} \times 4 = 3.$$

$$\frac{1}{8} \text{ of } \frac{5}{16} = \frac{1}{8} \times \frac{5}{16} = \frac{5}{128}.$$

EXAMPLE.—Multiply $\frac{3}{8}$ by 2.

SOLUTION.— $2 \times \frac{3}{8} = \frac{3 \times 2}{8} = \frac{6}{8} = \frac{3}{4}$. Ans.

Or $2 \times \frac{3}{8} = \frac{3}{8 \div 2} = \frac{3}{4}$. Ans.

115. EXAMPLE.—What is the product of $\frac{4}{16}$ and $\frac{7}{8}$?

SOLUTION.— $\frac{4}{16} \times \frac{7}{8} = \frac{4 \times 7}{16 \times 8} = \frac{28}{128} = \frac{7}{32}$. Ans.

or, by cancelation, $\frac{\cancel{4} \times 7}{\underset{4}{\cancel{16}} \times 8} = \frac{7}{4 \times 8} = \frac{7}{32}$. Ans.

116. EXAMPLE.—What is $\frac{4}{8}$ of $\frac{3}{4}$ of $\frac{16}{32}$?

SOLUTION.— $\frac{\cancel{4} \times 3 \times \cancel{16}}{8 \times \cancel{4} \times \underset{2}{\cancel{32}}} = \frac{3}{8 \times 2} = \frac{3}{16}$. Ans.

117. Example.—What is the product of $9\frac{3}{4}$ and $5\frac{5}{8}$?

Solution.—$9\frac{3}{4} = \frac{39}{4}$; $5\frac{5}{8} = \frac{45}{8}$.

$$\frac{39}{4} \times \frac{45}{8} = \frac{39 \times 45}{4 \times 8} = \frac{1755}{32} = 54\frac{27}{32}. \quad \text{Ans.}$$

118. Example.—Multiply $15\frac{7}{8}$ by 3.

Solution.—

$$\begin{array}{r} 15\frac{7}{8} \\ 3 \\ \hline 47\frac{5}{8} \end{array} \quad \text{or} \quad \begin{array}{r} 15\frac{7}{8} \\ 3 \\ \hline 45 \end{array} + \frac{21}{8} = 45 + 2\frac{5}{8} = 47\frac{5}{8}. \quad \text{Ans.}$$

119. Rule.—I. *Divide the product of the numerators by the product of the denominators. All factors common to the numerators and denominators should first be cast out by cancelation.*

II. *To multiply one mixed number by another, reduce them both to improper fractions.*

III. *To multiply a mixed number by a whole number, first multiply the fractional part by the multiplier, and if the product is an improper fraction, reduce it to a mixed number, and add the whole number part to the product of the multiplier and whole number.*

EXAMPLES FOR PRACTICE.

120. Find the product of

(*a*) $7 \times \frac{3}{16}$.	Ans.	(*a*) $1\frac{5}{16}$.
(*b*) $14 \times \frac{5}{16}$.		(*b*) $4\frac{3}{8}$.
(*c*) $\frac{12}{16} \times \frac{5}{16}$.		(*c*) $\frac{15}{64}$.
(*d*) $\frac{11}{17} \times 4$.		(*d*) $2\frac{10}{17}$.
(*e*) $\frac{17}{16} \times 7$.		(*e*) $7\frac{7}{16}$.
(*f*) $17\frac{12}{14} \times 7$.		(*f*) 125.
(*g*) $\frac{105}{224} \times 32$.		(*g*) 15.
(*h*) $\frac{15}{28} \times 14$.		(*h*) $7\frac{1}{2}$.

DIVISION OF FRACTIONS.

121. In *division* of fractions it is not necessary to *reduce* the *fractions* to fractions having a *common denominator*.

122. Dividing the *numerator* or *multiplying* the *denominator*, *divides* the fraction.

Example.—Divide $\frac{6}{8}$ by 3.

Solution.—When *dividing* the *numerator*, we have

$$\frac{6}{8} \div 3 = \frac{6 \div 3}{8} = \frac{2}{8} = \frac{1}{4}. \quad \text{Ans.}$$

When *multiplying* the *denominator*, we have

$$\tfrac{6}{8} \div 3 = \frac{6}{8 \times 3} = \tfrac{6}{24} = \tfrac{1}{4}. \quad \text{Ans.}$$

Example.—Divide $\frac{3}{16}$ by 2.

Solution.— $$\tfrac{3}{16} \div 2 = \frac{3}{16 \times 2} = \tfrac{3}{32}. \quad \text{Ans.}$$

Example.—Divide $\frac{14}{32}$ by 7.

Solution.— $$\tfrac{14}{32} \div 7 = \frac{14 \div 7}{32} = \tfrac{2}{32} = \tfrac{1}{16}. \quad \text{Ans.}$$

123. To **invert** a fraction is to *turn* it *upside down*; that is, make the *numerator* and *denominator change places*.

Invert $\frac{3}{4}$ and it becomes $\frac{4}{3}$.

124. Example.—Divide $\frac{9}{16}$ by $\frac{3}{16}$.

Solution.—1. The fraction $\frac{3}{16}$ is contained in $\frac{9}{16}$, 3 times, for the *denominators* are the same, and one *numerator* is *contained* in the other 3 times. 2. If we now *invert* the *divisor* $\frac{3}{16}$, and *multiply*, the solution is

$$\frac{9}{16} \times \frac{16}{3} = \frac{\overset{3}{\cancel{9}} \times \cancel{16}}{\cancel{16} \times \cancel{3}} = 3. \quad \text{Ans.}$$

This brings the *same quotient* as in the first case.

125. Example.—Divide $\frac{3}{8}$ by $\frac{1}{4}$.

Solution.—We cannot divide $\frac{3}{8}$ by $\frac{1}{4}$, as in the first case above, for the *denominators* are *not* the same, therefore, we must solve as in the second case.

$$\tfrac{3}{8} \div \tfrac{1}{4} = \tfrac{3}{8} \times \tfrac{4}{1} = \frac{3 \times \cancel{4}}{\underset{2}{\cancel{8}} \times 1} = \frac{3}{2} \text{ or } 1\tfrac{1}{2}. \quad \text{Ans.}$$

126. Example.—Divide 5 by $\frac{10}{16}$.

Solution.— $\frac{10}{16}$ inverted becomes $\frac{16}{10}$.

$$5 \times \frac{16}{10} = \frac{\cancel{5} \times \overset{8}{\cancel{16}}}{\underset{2}{\cancel{10}}} = 8. \quad \text{Ans.}$$

127. Example.—How many times is $3\frac{3}{4}$ contained in $7\frac{7}{16}$?

Solution.— $3\frac{3}{4} = \frac{15}{4}$; $7\frac{7}{16} = \frac{119}{16}$.

$\frac{15}{4}$ inverted equals $\frac{4}{15}$.

$$\frac{119}{16} \times \frac{4}{15} = \frac{119 \times \cancel{4}}{\underset{4}{\cancel{16}} \times 15} = \frac{119}{60} = 1\tfrac{59}{60}. \quad \text{Ans.}$$

128. Rule.—*Invert the divisor, and proceed as in multiplication.*

129. We have learned that a line placed between two numbers indicates that the number above the line is to be divided by the number below it. Thus, $\frac{18}{3}$ shows that 18 is to be divided by 3. This is also true if a fraction or a fractional expression be placed above or below a line.

$\dfrac{9}{\frac{3}{8}}$ means that 9 is to be divided by $\frac{3}{8}$; $\dfrac{3 \times 7}{\frac{8+4}{16}}$ means that 3 × 7 is to be divided by the value of $\frac{8+4}{16}$.

$\dfrac{\frac{1}{4}}{\frac{3}{8}}$ is the same as $\frac{1}{4} \div \frac{3}{8}$.

It will be noticed that there is a heavy line between the 9 and the $\frac{3}{8}$. This is necessary, since otherwise there would be nothing to show as to whether 9 was to be divided by $\frac{3}{8}$, or $\frac{9}{3}$ was to be divided by 8. Whenever a heavy line is used, as shown here, it indicates that *all above the line* is to be divided by *all below it.*

EXAMPLES FOR PRACTICE.

130. Divide

(*a*)	15 by $6\frac{3}{7}$.	Ans.	(*a*)	$2\frac{1}{3}$.
(*b*)	30 by $\frac{3}{4}$.		(*b*)	40.
(*c*)	172 by $\frac{4}{5}$.		(*c*)	215.
(*d*)	$\frac{14}{18}$ by $1\frac{7}{16}$.		(*d*)	$\frac{112}{207}$.
(*e*)	$\frac{105}{6}$ by $14\frac{2}{3}$.		(*e*)	$1\frac{17}{88}$.
(*f*)	$\frac{142}{27}$ by $17\frac{1}{3}$.		(*f*)	$\frac{71}{234}$.
(*g*)	$\frac{14}{18}$ by $\frac{145}{72}$.		(*g*)	$\frac{56}{145}$.
(*h*)	$\frac{132}{15}$ by $72\frac{1}{5}$.		(*h*)	$\frac{44}{361}$.

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131. Whenever an expression like one of the three following is obtained, it may always be simplified by transposing the denominator from *above* to *below* the line, or from *below* to *above*, as the case may be, taking care however to indicate that the denominator when so transferred is a multiplier.

1. $\dfrac{\frac{3}{4}}{9} = \dfrac{3}{9 \times 4} = \frac{3}{36} = \frac{1}{12}$; for, regarding the fraction above the heavy line as the numerator of a fraction whose denominator is 9, $\dfrac{\frac{3}{4} \times 4}{9 \times 4} = \dfrac{3}{9 \times 4}$, as before.

2. $\frac{9}{\frac{3}{4}} = \frac{9 \times 4}{3} = 12.$ The proof is the same as in the first case.

3. $\frac{\frac{5}{9}}{\frac{3}{4}} = \frac{5 \times 4}{3 \times 9} = \frac{20}{27}$; for, regarding $\frac{5}{9}$ as the numerator of a fraction whose denominator is $\frac{3}{4}$, $\frac{\frac{5}{9} \times 9}{\frac{3}{4} \times 9} = \frac{5}{\frac{3 \times 9}{4}}$; and
$\frac{\frac{5}{\frac{3 \times 9}{4}} \times 4}{\frac{3 \times 9}{4} \times 4} = \frac{5 \times 4}{3 \times 9} = \frac{20}{27}$, as above.

This principle may be used to great advantage in cases like $\frac{\frac{1}{4} \times 310 \times \frac{27}{12} \times 72}{40 \times 4\frac{1}{2} \times 5\frac{1}{6}}$. Reducing the mixed numbers to fractions, the expression becomes $\frac{\frac{1}{4} \times 310 \times \frac{27}{12} \times 72}{40 \times \frac{9}{2} \times \frac{31}{6}}$. Now transferring the denominators of the fractions and canceling,

$$\frac{1 \times 310 \times 27 \times 72 \times 2 \times 6}{40 \times 9 \times 31 \times 4 \times 12} = \frac{1 \times \overset{\cancel{10}}{\cancel{310}} \times \overset{3}{\cancel{27}} \times \overset{\overset{3}{\cancel{6}}}{\cancel{72}} \times \cancel{2} \times \overset{3}{\cancel{6}}}{\underset{\underset{\cancel{2}}{\cancel{4}}}{\cancel{40}} \times \cancel{9} \times \cancel{31} \times \underset{2}{\cancel{4}} \times \cancel{12}} =$$

$\frac{27}{2} = 13\frac{1}{2}.$

Greater exactness in results can usually be obtained by using this principle than can be obtained by reducing the fractions to decimals. The principle, however, should not be employed *if a sign of addition or subtraction occurs either above or below the dividing line.*

ARITHMETIC.

(SECTION 3.)

DECIMALS.

Remark.—A knowledge of decimals is of the utmost importance to all who are required to make calculations of any kind. The subject is easy to learn, and for this reason the student is somewhat inclined to study it too hastily, the result being that he afterwards has trouble that might have been entirely avoided had he given the text the proper attention in the beginning. Decimals are much easier to use than common fractions, which they replace; at the same time it is frequently more expedient to use common fractions in certain operations, and, hence, they cannot be wholly dispensed with. Particular attention should be paid to the rules for multiplication and division—especially to the locating of the decimal point—and to the operations of changing a common fraction to a decimal and vice versa.

132. **Decimals** are *tenth* fractions; that is, the parts of a unit are expressed on the scale of ten, as *tenths*, *hundredths*, *thousandths*, etc.

133. The *denominator*, which is always ten or a multiple of ten, as 10, 100, 1,000, etc., is *not* expressed as it would be in common fractions, by writing it under the *numerator*, with a line between them; as, $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$. The denominator is always understood, the numerator consisting of the figures on the right of the *unit* figure. In order to distinguish

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the unit figure, a period (.), called the **decimal point,** is placed between the unit figure and the next figure on the right. The decimal point may be regarded in two ways: first, as indicating that the number on the right is the numerator of a fraction whose denominator is 10, 100, 1,000, etc.; and, second, as a part of the Arabic system of notation, each figure on the right being 10 times as large as the next succeeding figure, and 10 times as small as the next preceding figure, serving merely to point out the unit figure.

134. The *reading* of a *decimal number* depends upon the *number* of *decimal places* in it, or the *number* of *figures* to the *right* of the unit figure.

The first figure to the right of the unit figure expresses *tenths.*

The second figure to the right of the unit figure expresses *hundredths.*

The third figure to the right of the unit figure expresses *thousandths.*

The fourth figure to the right of the unit figure expresses *ten-thousandths.*

The fifth figure to the right of the unit figure expresses *hundred-thousandths.*

The sixth figure to the right of the unit figure expresses *millionths.*

Thus:

$.3 = \frac{3}{10} =$ 3 tenths.
$.03 = \frac{3}{100} =$ 3 hundredths.
$.003 = \frac{3}{1000} =$ 3 thousandths.
$.0003 = \frac{3}{10000} =$ 3 ten-thousandths.
$.00003 = \frac{3}{100000} =$ 3 hundred-thousandths.
$.000003 = \frac{3}{1000000} =$ 3 millionths.

The first figure to the right of the unit figure is called the *first* **decimal place;** the second figure, the *second decimal place*, etc. We see in the above that the *number* of *decimal places* in a decimal equals the *number of ciphers* to the *right* of the figure 1 in the *denominator* of its *equivalent fraction.* This fact kept in mind will be of much assistance in reading and writing decimals.

Whatever may be written to the *left* of a *decimal point* is a whole number. The decimal point affects only the figures to its *right.*

When a *whole number* and *decimal* are written together, the expression is a *mixed number.* Thus, 8.12 and 17.25 are mixed numbers.

The relation of decimals and whole numbers to each other is clearly shown by the following table :

hundreds of millions.	tens of millions.	millions.	hundreds of thousands.	tens of thousands.	thousands.	hundreds.	tens.	units.	decimal point.	tenths.	hundredths.	thousandths.	ten-thousandths.	hundred-thousandths.	millionths.	ten-millionths.	hundred-millionths.
9	8	7	6	5	4	3	2	1	.	2	3	4	5	6	7	8	9

The figures to the *left* of the *decimal point* represent *whole numbers;* those to the *right* are *decimals.*

In *both* the decimals and whole numbers, the *units* place is made the *starting point* of notation and numeration. The *decimals decrease* on the scale of *ten* to the *right*, and the *whole* numbers *increase* on the scale of *ten* to the *left.* The *first* figure to the *left* of units is *tens*, and the *first* figure to the *right* of units is *tenths.* The *second* figure to the *left* of units is *hundreds*, and the *second* figure to the *right* is *hundredths.* The *third* figure to the *left* is *thousands*, and the *third* to the *right* is *thousandths*, and so on ; the *whole* numbers on the *left* and the *decimals* on the *right.* The figures equally distant from units place correspond in name. The *decimals* have the ending *ths*, which distinguishes them from *whole* numbers. The following is the numeration of the number in the above table : Nine hundred eighty-seven million, six hundred fifty-four thousand, three hundred twenty-one, and twenty-three million, four hundred fifty-six thousand, seven hundred eightv-nine hundred millionths.

The *decimals* increase to the *left*, on the scale of *ten*, the same as *whole* numbers; for, beginning at, say, 4-*thousandths*, in the table, the next figure to the left is *hundredths*, which is ten times as great, and the next *tenths*, or ten times the *hundredths*, and so on through both decimals and whole numbers.

135. *Annexing* or *taking away* a *cipher* at the *right* of a *decimal* does *not* affect its value.

.5 is $\frac{5}{10}$; .50 is $\frac{50}{100}$, but $\frac{5}{10} = \frac{50}{100}$; therefore, $.5 = .50$.

136. *Inserting* a *cipher* between a *decimal* and the *decimal point divides* the decimal by 10.

$.5 = \frac{5}{10}$; $\frac{5}{10} \div 10 = \frac{5}{100} = .05$.

137. *Taking away* a *cipher* from the *left* of a *decimal multiplies* the *decimal* by 10.

$.05 = \frac{5}{100}$; $\frac{5}{100} \times 10 = \frac{5}{10} = .5$.

138. In some cases it is convenient to express a mixed decimal fraction in the form of a common (improper) fraction. To do so it is only necessary to write the entire number, omitting the decimal point, as the numerator of the fraction, and the denominator of the decimal part as the denominator of the fraction. Thus, $127.483 = \frac{127483}{1000}$; for, $127.483 = 127\frac{483}{1000} = \frac{127000+483}{1000} = \frac{127483}{1000}$.

ADDITION OF DECIMALS.

139. The only respect in which *addition* of *decimals* differs from *addition* of *whole numbers*, is that while the unit figures are placed under each other in both cases, the right-hand figures are not necessarily in line when adding decimals.

Whole numbers begin at units and increase on the scale of 10, to the left. Decimals decrease on the scale of 10, to the right. Whole numbers are to the left of the decimal point and decimals are to the right of it. In whole numbers the *right-hand* side of a column of figures to be added, must be in line, and in decimals, the *left-hand* side must be in line, which brings the decimal points directly under each other.

	whole numbers		*decimals*		*mixed numbers*	
	342		.342		342.032	
	4234		.4234		4234.5	
	26		.26		26.6782	
	3		.03		3.06	
sum	4605	Ans.	1.0554	Ans.	4606.2702	Ans.

140. A decimal, as .342, ought really to be expressed as 0.342, but it is quite customary to omit the cipher on the left of the decimal point, though many authors use it.

Example.—What is the sum of 242, .36, 118.725, 1.005, 6, and 100.1?

Solution.—

$$\begin{array}{lr@{}l} & 242 & . \\ & & .36 \\ & 118 & .725 \\ & 1 & .005 \\ & 6 & . \\ & 100 & .1 \\ \hline \textit{sum} & 468 & .190 \text{ Ans.} \end{array}$$

141. Rule.—*Place the numbers to be added so that the decimal points will be directly under each other. Add as in whole numbers, and place the decimal point in the sum directly under the decimal points above.*

EXAMPLES FOR PRACTICE.

142. Find the sum of

		Ans.
(*a*)	.2143, .105, 2.3042, and 1.1417.	(*a*) 3.7652.
(*b*)	783.5, 21.473, .2101, and .7816.	(*b*) 805.9647.
(*c*)	21.781, 138.72, 41.8738, .72, and 1.413.	(*c*) 204.5078.
(*d*)	.3724, 104.15, 21.417, and 100.042.	(*d*) 225.9814.
(*e*)	200.172, 14.105, 12.1465, .705, and 7.2.	(*e*) 234.3285.
(*f*)	1,427.16, .244, .32, .032, and 10.0041.	(*f*) 1,437.7601.
(*g*)	2,473.1, 41.65, .7243, 104.067, and 21.073.	(*g*) 2,640.6143.
(*h*)	4,107.2, .00375, 21.716, 410.072, and .0345.	(*h*) 4,539.02625.

SUBTRACTION OF DECIMALS.

143. For the same reason as in addition of decimals, the *left-hand* figures of *decimal numbers* are placed in line and the *decimal points* under each other.

Example.—Subtract .132 from .3063.

Solution.—

minuend	.3063
subtrahend	.132
difference	.1743 Ans.

144. **Example.**—What is the difference between 7.895 and .725?

Solution.—

minuend	7.895
subtrahend	.725
difference	7.170 or 7.17 Ans.

145. **Example.**—Subtract .625 from 11.

Solution.—

minuend	11.000
subtrahend	.625
difference	10.375 Ans.

146. **Rule.**—*Place the subtrahend under the minuend, so that the decimal points will be directly under each other. Subtract, as in whole numbers, and place the decimal point in the remainder, directly under the decimal points above.*

When the figures in the decimal part of the subtrahend extend beyond those in the minuend, place ciphers in the minuend above them, and subtract as before.

EXAMPLES FOR PRACTICE.

147. From

		Ans.
(*a*)	407.385 take 235.0004.	(*a*) 172.3846.
(*b*)	22.718 take 1.7042.	(*b*) 21.0138.
(*c*)	1,368.17 take 13.6817.	(*c*) 1,354.4883.
(*d*)	70.00017 take 7.000017.	(*d*) 63.000153.
(*e*)	630.630 take .6304.	(*e*) 629.9996.
(*f*)	421.73 take 217.162.	(*f*) 204.568.
(*g*)	1.000014 take .00001.	(*g*) 1.000004.
(*h*)	.783652 take .542314.	(*h*) .241338.

MULTIPLICATION OF DECIMALS.

148. In multiplication of decimals, we do not place the decimal points directly under each other as in addition and subtraction. We pay no attention for the time being to the decimal points. Place the multiplier under the multiplicand, so that the *right-hand* figure of the one is under the *right-hand* figure of the other, and proceed exactly as in multiplication of whole numbers. After multiplying, *count the number of decimal places in both multiplicand and multiplier, and point off the same number in the product.*

Example.—Multiply .825 by 13.

Solution.—
$$\begin{array}{rl} \textit{multiplicand} & .825 \\ \textit{multiplier} & 13 \\ \hline & 2475 \\ & 825 \\ \hline \textit{product} & 10.725 \end{array}$$
Ans.

In this example there are three decimal places in the multiplicand and none in the multiplier; therefore, 3 decimal places are pointed off in the product.

149. Example.—What is the product of 426 and the decimal .005 ?

Solution.—
$$\begin{array}{rl} \textit{multiplicand} & 426 \\ \textit{multiplier} & .005 \\ \hline \textit{product} & 2.130 \text{ or } 2.13 \end{array}$$
Ans.

In this example there are 3 decimal places in the multiplier and none in the multiplicand; therefore, 3 decimal places are pointed off in the product.

150. It is *not* necessary to multiply by the ciphers on the *left* of a *decimal;* they merely determine the number of decimal places. Ciphers to the *right* of a decimal should be removed, as they only make more figures to deal with, and do not change the value.

151. Example.—Multiply 1.205 by 1.15.

Solution.—
$$\begin{array}{rr} \textit{multiplicand} & 1.205 \\ \textit{multiplier} & 1.15 \\ \hline & 6025 \\ & 1205 \\ & 1205 \\ \hline \textit{product} & 1.38575 \end{array}$$
Ans.

In this example there are 3 decimal places in the multiplicand, and 2 in the multiplier; therefore, 3 + 2, or 5, decimal places must be pointed off in the product.

152. Example.—Multiply .232 by .001.

Solution.— *multiplicand* .232
multiplier .001
product .000232 Ans.

In this example we multiply the multiplicand by the digit in the multiplier, which makes 232 in the product, but since there are 3 decimal places in each, the multiplier and the multiplicand, we must prefix 3 ciphers to the 232, to make 3 + 3, or 6, decimal places in the product.

153. Rule.—*Place the multiplier under the multiplicand, disregarding the position of the decimal points. Multiply as in whole numbers, and in the product point off as many decimal places as there are decimal places in both multiplier and multiplicand, prefixing ciphers if necessary.*

EXAMPLES FOR PRACTICE.

154. Find the product of

	Ans.
(*a*) .000492 × 4.1418.	(*a*) .0020377656.
(*b*) 4,003.2 × 1.2.	(*b*) 4,803.84.
(*c*) 78.6531 × 1.03.	(*c*) 81.012693.
(*d*) .3685 × .042.	(*d*) .015477.
(*e*) 178,352 × .01.	(*e*) 1,783.52.
(*f*) .00045 × .0045.	(*f*) .000002025.
(*g*) .714 × .00002.	(*g*) .00001428.
(*h*) .00004 × .008.	(*h*) .00000032.

DIVISION OF DECIMALS.

155. In division of decimals we pay *no* attention to the decimal point until *after* the division is performed. The *number of decimal places* in the *dividend must equal* (*be made to equal by annexing ciphers*) *the number of decimal places in the divisor.* Divide exactly as in whole numbers. *Subtract the number of decimal places in the divisor from the number of decimal places in the dividend, and point off as many decimal* *places in the quotient as there are units in the remainder thus found.*

EXAMPLE.—Divide .625 by 25.

```
                 divisor dividend quotient
SOLUTION.—         2 5 ) .6 2 5 ( .0 2 5   Ans.
                             5 0
                             ---
                             1 2 5
                             1 2 5
                             -----
                 remainder       0
```

In this example there are no decimal places in the divisor, and 3 decimal places in the dividend; therefore, there are 3 minus 0, or 3, decimal places in the quotient. One cipher has to be prefixed to the 25, to make the 3 decimal places.

156. EXAMPLE.—Divide 6.035 by .05.

```
                 divisor  dividend  quotient
SOLUTION.—        .0 5 ) 6.0 3 5 ( 1 2 0.7   Ans.
                         5
                         -
                         1 0
                         1 0
                         ---
                             3 5
                             3 5
                             ---
                 remainder     0
```

In this example we divide by 5, as if the cipher were not before it. There is one more decimal place in the dividend than in the divisor; therefore, one decimal place is pointed off in the quotient.

157. EXAMPLE.—Divide .125 by .005.

```
                 divisor dividend quotient
SOLUTION.—       .0 0 5 ) .1 2 5 ( 2 5   Ans.
                             1 0
                             ---
                               2 5
                               2 5
                               ---
                 remainder       0
```

In this example there are the same number of decimal places in the dividend as in the divisor; therefore, the quotient has no decimal places, and is a whole number.

158. EXAMPLE.—Divide 326 by .25.

divisor *dividend* *quotient*
SOLUTION.— .25)326.00(1304 Ans.
25
76
75
100
100
remainder 0

In this problem two ciphers were annexed to the dividend, to make the number of decimal places equal to the number in the divisor. The quotient is a whole number.

159. EXAMPLE.—Divide .0025 by 1.25.

SOLUTION.— 1.25).00250(.002 Ans.
250
remainder 0

EXPLANATION.—In this example we are to divide .0025 by 1.25. Consider the dividend as a whole number, or 25 (disregarding the two ciphers at its left, for the present); also, consider the divisor as a whole number, or 125. It is clearly evident that the dividend 25 will not contain the divisor 125; we must, therefore, annex one cipher to the 25, thus making the dividend 250. 125 is contained twice in 250, so we place the figure 2 in the quotient. In pointing off the decimal places in the quotient, it must be remembered that there were only four decimal places in the dividend; but one cipher was annexed, thereby making 4 + 1, or 5, decimal places. Since there are 5 decimal places in the dividend and 2 decimal places in the divisor, we must point off 5 − 2, or 3, decimal places in the quotient. In order to point off 3 decimal places, two ciphers must be prefixed to the figure 2, thereby making .002 the quotient. It is not necessary to consider the ciphers at the left of a decimal when dividing, except when determining the position of the decimal point in the quotient.

160. **Rule.—I.** *Place the divisor to the left of the dividend, and proceed as in division of whole numbers; in the* *quotient, point off as many decimal places as the number of decimal places in the dividend exceeds those in the divisor, prefixing ciphers to the quotient, if necessary.*

II. *If in dividing one number by another there be a remainder, the remainder can be placed over the divisor, as a fractional part of the quotient, but it is generally better to annex ciphers to the remainder, and continue dividing until there are 3 or 4 decimal places in the quotient, and then if there still be a remainder, terminate the quotient by the plus sign* (+), *which shows that it can be carried further.*

161. Example.—What is the quotient of 199 divided by 15?

Solution.— $15)199(13+\frac{4}{15}$ Ans.

```
           15
           --
           49
           45
           --
remainder   4
```

Or, 15)199.000(13.266+ Ans.

```
           15
           --
           49
           45
           ---
            40
            30
            ---
            100
             90
            ---
            100
             90
            ---
remainder    10
```

$13\frac{4}{15} = 13.266+$

$\frac{4}{15} = .266+$

162. It frequently happens, as in the above example, that the division will never terminate. In such cases, decide to how many decimal places the division is to be carried, and carry the work one place further. If the last figure of the quotient thus obtained is 5 or a greater number, increase the preceding figure by 1, and write after it the minus sign (−), thus indicating that the quotient is not quite as large as indicated; if the figure thus obtained is less than 5, write the plus sign (+) after the quotient, thus indicating that the number is slightly greater than as indicated. In the last example, had it been desired to obtain the answer correct to four decimal places, the work would have been carried to five places, obtaining 13.26666, and the answer would have been given as 13.2667—. This remark applies to any other calculation involving decimals, when it is desired to omit some of the figures in the decimal. Thus, if it is desired to retain three decimal places in the number .2471253, it would be expressed as .247 +; if it was desired to retain five decimal places, it would be expressed as .24713—. Both the + and — signs are frequently omitted; they are seldom used outside of Arithmetic, except in exact calculations, when it is desired to call particular attention to the fact that the result obtained is not *quite* exact.

EXAMPLES FOR PRACTICE.

163. Divide

(*a*)	101.6688 by 2.36.	Ans.	(*a*)	43.08.
(*b*)	187.12264 by 123.107.		(*b*)	1.52.
(*c*)	.08 by .008.		(*c*)	10.
(*d*)	.0003 by 3.75.		(*d*)	.00008.
(*e*)	.0144 by .024.		(*e*)	.6.
(*f*)	.00375 by 1.25.		(*f*)	.003.
(*g*)	.004 by 400.		(*g*)	.00001.
(*h*)	.4 by .008.		(*h*)	50.

TO REDUCE A FRACTION TO A DECIMAL.

164. EXAMPLE.—$\frac{3}{4}$ equals what decimal?

SOLUTION.— $\dfrac{4\,)\,3.0\,0}{.7\,5}$, or $\frac{3}{4} = .75$. Ans.

EXAMPLE.—What decimal is equivalent to $\frac{7}{8}$?

SOLUTION.—
$$\begin{array}{r@{}l} 8\,)\,7.0\,0\,0\,(&.8\,7\,5 \\ \underline{6\,4}& \\ 6\,0& \\ \underline{5\,6}& \quad \text{or } \tfrac{7}{8} = .875.\ \text{Ans.} \\ 4\,0& \\ \underline{4\,0}& \\ 0& \end{array}$$

165. Rule.—*Annex ciphers to the numerator and divide by the denominator. Point off as many decimal places in the quotient as there are ciphers annexed.*

EXAMPLES FOR PRACTICE.

166. Reduce the following common fractions to decimals:

(*a*)	$\frac{15}{32}$.	Ans.	(*a*)	.46875.
(*b*)	$\frac{7}{8}$.		(*b*)	.875.
(*c*)	$\frac{21}{32}$.		(*c*)	.65625.
(*d*)	$\frac{51}{64}$.		(*d*)	.796875.
(*e*)	$\frac{4}{25}$.		(*e*)	.16.
(*f*)	$\frac{5}{8}$.		(*f*)	.625.
(*g*)	$\frac{10}{200}$.		(*g*)	.05.
(*h*)	$\frac{4}{1000}$.		(*h*)	.004.

---

167. To reduce inches to decimal parts of a foot:

EXAMPLE.—What decimal part of a foot is 9 inches?

SOLUTION.—Since there are 12 inches in one foot, 1 inch is $\frac{1}{12}$ of a foot, and 9 inches is $9 \times \frac{1}{12}$ or $\frac{9}{12}$ of a foot. This, reduced to a decimal by the above rule, shows what decimal part of a foot 9 inches is.

12) 9.00 (.75 of a foot. Ans.
84
60
60
0

168. Rule I.—*To reduce inches to decimal parts of a foot, divide the number of inches by 12.*

II. *Should the resulting decimal be an unending one and it is desired to terminate the division at some point, say, the fourth decimal place, carry the division one place further, and if the fifth figure is 5 or greater, increase the fourth figure by 1. Omit the signs + and —.*

EXAMPLES FOR PRACTICE.

169. Reduce to the decimal part of a foot:

(*a*)	3 in.	Ans.	(*a*)	.25.
(*b*)	4½ in.		(*b*)	.375.
(*c*)	5 in.		(*c*)	.4167.
(*d*)	6⅝ in.		(*d*)	.5521.
(*e*)	11 in.		(*e*)	.9167.

TO REDUCE A DECIMAL TO A FRACTION.

170. Example.—Reduce .125 to a fraction.

Solution.— $.125 = \frac{125}{1000} = \frac{5}{40} = \frac{1}{8}$. Ans.

Example.—Reduce .875 to a fraction.

Solution.— $.875 = \frac{875}{1000} = \frac{35}{40} = \frac{7}{8}$. Ans.

171. Rule.—*Under the figures of the decimal, place 1 with as many ciphers at its right as there are decimal places in the decimal, and reduce the resulting fraction to its lowest terms by dividing both numerator and denominator by the same number.*

EXAMPLES FOR PRACTICE.

172. Reduce the following to common fractions:

	Ans.
(*a*) .125.	(*a*) $\frac{1}{8}$.
(*b*) .625.	(*b*) $\frac{5}{8}$.
(*c*) .3125.	(*c*) $\frac{5}{16}$.
(*d*) .04.	(*d*) $\frac{1}{25}$.
(*e*) .06.	(*e*) $\frac{3}{50}$.
(*f*) .75.	(*f*) $\frac{3}{4}$.
(*g*) .15625.	(*g*) $\frac{5}{32}$.
(*h*) .875.	(*h*) $\frac{7}{8}$.

173. To express a decimal approximately as a fraction having a given denominator:

174. Example.—Express .5827 in 64ths.

Solution.— $.5827 \times \frac{64}{64} = \frac{37.2928}{64}$, say $\frac{37}{64}$.

Hence, $.5827 = \frac{37}{64}$, nearly. Ans.

Example.—Express .3917 in 12ths.

Solution.— $.3917 \times \frac{12}{12} = \frac{4.7004}{12}$, say $\frac{5}{12}$.

Hence, $.3917 = \frac{5}{12}$, nearly. Ans.

175. Rule.—*Reduce 1 to a fraction having the given denominator. Multiply the given decimal by the fraction so obtained, and the result will be the fraction required.*

EXAMPLES FOR PRACTICE.

176. Express

	Ans.
(*a*) .625 in 8ths.	(*a*) $\frac{5}{8}$.
(*b*) .3125 in 16ths.	(*b*) $\frac{5}{16}$.
(*c*) .15625 in 32ds.	(*c*) $\frac{5}{32}$.
(*d*) .77 in 64ths.	(*d*) $\frac{49}{64}$.
(*e*) .81 in 48ths.	(*e*) $\frac{39}{48}$.
(*f*) .923 in 96ths.	(*f*) $\frac{89}{96}$.

177. The sign for dollars is $. It is read dollars. $25 is read 25 dollars.

Since there are 100 cents in a dollar, one cent is 1-one-hundredth of a dollar; the first two figures of a decimal part of a dollar represent *cents*. Since a mill is $\frac{1}{10}$ of a cent, or $\frac{1}{1000}$ of a dollar, the third figure represents mills.

Thus, $25.16 is read twenty-five dollars and sixteen cents; $25.168 is read twenty-five dollars, sixteen cents and eight mills.

178. The **vinculum** ——, **parenthesis** ( ), **bracket** [ ], and **brace** { } are called **symbols of aggregation,** and are used to include numbers which are to be considered together; thus, $13 \times \overline{8 - 3}$, or $13 \times (8 - 3)$, shows that 3 is to be taken from 8 before multiplying by 13.

$$13 \times (8 - 3) = 13 \times 5 = 65. \quad \text{Ans.}$$
$$13 \times \overline{8 - 3} = 13 \times 5 = 65. \quad \text{Ans.}$$

When the vinculum or parenthesis is not used, we have

$$13 \times 8 - 3 = 104 - 3 = 101. \quad \text{Ans.}$$

179. In any series of numbers connected by the signs +, −, ×, and ÷, the operations indicated by the signs must be performed in order from left to right, *except* that no addition or subtraction may be performed if a sign of multiplication or division *follows* the number on the *right* of a sign of addition or subtraction, until the indicated multiplication or division has been performed. In all cases the sign of multiplication takes the precedence, the reason being that when two or more numbers or expressions are connected by the sign of multiplication, the numbers thus connected are regarded as factors of the product indicated, and not as separate numbers.

EXAMPLE.—What is the value of $4 \times 24 - 8 + 17$?

SOLUTION.—Performing the operations in order from left to right, $4 \times 24 = 96$; $96 - 8 = 88$; $88 + 17 = 105$. Ans.

180. EXAMPLE.—What is the value of the following expression: $1{,}296 \div 12 + 160 - 22 \times 3\frac{1}{2} = ?$

SOLUTION.—$1{,}296 \div 12 = 108$; $108 + 160 = 268$; here we cannot subtract 22 from 268 because the sign of multiplication *follows* 22; hence, multiplying 22 by $3\frac{1}{2}$, we get 77, and $268 - 77 = 191$. Ans.

Had the above expression been written $1,296 \div 12 + 160 - 22 \times 3\frac{1}{2} \div 7 + 25$, it would have been necessary to have divided $22 \times 3\frac{1}{2}$ by 7 before subtracting, and the final result would have been $22 \times 3\frac{1}{2} = 77$; $77 \div 7 = 11$; $268 - 11 = 257$; $257 + 25 = 282$. Ans. In other words, it is necessary to perform *all* of the multiplication or division included between the signs + and −, or − and +, before adding or subtracting. Also, had the expression been written $1,296 \div 12 + 160 - 24\frac{1}{2} \div 7 \times 3\frac{1}{2} + 25$, it would have been necessary to have multiplied $3\frac{1}{2}$ by 7 before dividing $24\frac{1}{2}$, since the sign of multiplication takes the precedence, and the final result would have been $3\frac{1}{2} \times 7 = 24\frac{1}{2}$; $24\frac{1}{2} \div 24\frac{1}{2} = 1$; $268 - 1 = 267$; $267 + 25 = 292$. Ans.

It likewise follows that if a succession of multiplication and division signs occurs, the indicated operations must not be performed in order, from left to right—the multiplication must be performed first. Thus, $24 \times 3 \div 4 \times 2 \div 9 \times 5 = \frac{1}{5}$. Ans. In order to obtain the same result that would be obtained by performing the indicated operations in order, from left to right, symbols of aggregation must be used. Thus, by using two vinculums, the last expression becomes $24 \times \overline{3 \div 4} \times \overline{2 \div 9} \times 5 = 20$, the same result that would be obtained by performing the indicated operations in order, from left to right.

EXAMPLES FOR PRACTICE.

181. Find the values of the following expressions:

(*a*)	$(8 + 5 - 1) \div 4$.	Ans.	(*a*)	3.
(*b*)	$5 \times 24 - 32$.		(*b*)	88.
(*c*)	$5 \times 24 \div 15$.		(*c*)	8.
(*d*)	$144 - 5 \times 24$.		(*d*)	24.
(*e*)	$(1,691 - 540 + 559) \div 3 \times 57$.		(*e*)	10.
(*f*)	$2,080 + 120 - 80 \times 4 - 1,670$.		(*f*)	210.
(*g*)	$\overline{(90 + 60} \div 25) \times 5 - 29$.		(*g*)	1.
(*h*)	$\overline{90 + 60} \div 25 \times 5$.		(*h*)	1.2.

ARITHMETIC.

(SECTION 4.)

PERCENTAGE.

182. **Percentage** is the process of calculating by *hundredths.*

183. The *term* **per cent.** is an abbreviation of the Latin words *per centum*, which mean *by the hundred.* A certain per cent. of a number is the number of hundredths of that number which is indicated by the number of units in the per cent. Thus, 6 per cent. of 125 is $125 \times \frac{6}{100} = 7.5$; 25 per cent. of 80 is $80 \times \frac{25}{100} = 20$; 43 per cent. of 432 pounds is $432 \times \frac{43}{100} = 185.76$ pounds.

184. The **sign** of per cent. is %, and is read *per cent.* Thus, 6% is read *six per cent.;* 12½% is read *twelve and one-half per cent.*, etc.

When expressing the per cent. of a number to use in calculations, it is customary to express it decimally instead of fractionally. Thus, instead of expressing 6%, 25%, and 43% as $\frac{6}{100}$, $\frac{25}{100}$, and $\frac{43}{100}$, it is usual to express them as .06, .25, and .43.

The following table will show how any per cent. can be expressed either as a decimal or as a fraction:

Per Cent.	Decimal.	Fraction.	Per Cent.	Decimal.	Fraction.
1%.....	.01	$\frac{1}{100}$	150 %....	1.50	$\frac{150}{100}$ or $1\frac{1}{2}$
2%.....	.02	$\frac{2}{100}$ or $\frac{1}{50}$	500 %....	5.00	$\frac{500}{100}$ or 5
5%.....	.05	$\frac{5}{100}$ or $\frac{1}{20}$	¼%....	.0025	$\frac{\frac{1}{4}}{100}$ or $\frac{1}{400}$
10%.....	.10	$\frac{10}{100}$ or $\frac{1}{10}$	½%....	.005	$\frac{\frac{1}{2}}{100}$ or $\frac{1}{200}$
25%.....	.25	$\frac{25}{100}$ or $\frac{1}{4}$	1½%....	.015	$\frac{1\frac{1}{2}}{100}$ or $\frac{3}{200}$
50%.....	.50	$\frac{50}{100}$ or $\frac{1}{2}$	8⅓%....	.08⅓	$\frac{8\frac{1}{3}}{100}$ or $\frac{1}{12}$
75%.....	.75	$\frac{75}{100}$ or $\frac{3}{4}$	12½%....	.125	$\frac{12\frac{1}{2}}{100}$ or $\frac{1}{8}$
100%.....	1.00	$\frac{100}{100}$ or 1	16⅔%....	.16⅔	$\frac{16\frac{2}{3}}{100}$ or $\frac{1}{6}$
125%.....	1.25	$\frac{125}{100}$ or $1\frac{1}{4}$	62½%....	.625	$\frac{62\frac{1}{2}}{100}$ or $\frac{5}{8}$

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185. The names of the different elements used in percentage are: the *base*, the *rate per cent.*, the *percentage*, the *amount*, and the *difference*.

186. The **base** is the number on which the per cent. is computed.

187. The **rate** is the number of hundredths of the base to be taken.

188. The **percentage** is the part, or number of *hundredths*, of the base indicated by the rate; or, the percentage is the result obtained by multiplying the base by the rate.

Thus, when it is stated that 7% of $25 is $1.75, $25 is the base, 7% is the rate, and $1.75 is the percentage.

189. The **amount** is the sum of the base and percentage.

190. The **difference** is the remainder obtained by subtracting the percentage from the base.

Thus, if a man has $180, and he earns 6% more, he will have, altogether, $\$180 + \$180 \times .06$, or $\$180 + \$10.80 = \$190.80$. Here $180 is the base; 6%, the rate; $10.80, the percentage, and $190.80, the *amount*.

Again, if an engine of 125 horsepower uses 16% of it in overcoming friction and other resistances, the amount left for obtaining useful work is $125 - 125 \times .16 = 125 - 20 = 105$ horsepower. Here 125 is the base; 16%, the rate; 20, the percentage, and 105, the *difference*.

191. From the foregoing it is evident that to find the percentage, the base must be multiplied by the rate. Hence, the following

Rule.—*To find the percentage, multiply the base by the rate expressed decimally.*

EXAMPLE.—Out of a lot of 300 bushels of apples, 76% were sold. How many bushels were sold?

SOLUTION.—76%, the rate, expressed decimally, is .76; the base is 300; hence, the number of bushels sold, or the percentage, is by the above rule,

$$300 \times .76 = 228 \text{ bushels. Ans.}$$

Expressing the rule as a

Formula, *percentage = base × rate.*

192. When the percentage and rate are given, the base may be found by dividing the percentage by the rate. For, suppose that 12 is 6%, or $\frac{6}{100}$, of some number; then, 1%, or $\frac{1}{100}$, of the number, is 12 ÷ 6, or 2. Consequently, if 2 = 1%, or $\frac{1}{100}$, 100%, or $\frac{100}{100}$, = 2 × 100 = 200. But, since the same result may be arrived at by dividing 12 by .06, for 12 ÷ .06 = 200, it follows that

Rule.—*When the percentage and rate are given, to find the base, divide the percentage by the rate, expressed decimally.*

Formula, *base* = *percentage* ÷ *rate.*

Example.—Bought a certain number of bushels of apples and sold 76% of them. If I sold 228 bushels, how many bushels did I buy?

Solution.—Here 228 is the percentage, and 76%, or .76, is the rate; hence, applying the rule,

228 ÷ .76 = 300 bushels. Ans.

193. When the base and percentage are given, to find the rate, the rate may be found, expressed decimally, by dividing the percentage by the base. For, suppose that it is desired to find what per cent. 12 is of 200. 1% of 200 is 200 × .01 = 2. Now, if 1% is 2, 12 is evidently as many per cent. as the number of times that 2 is contained in 12, or 12 ÷ 2 = 6%. But the same result may be obtained by dividing 12, the percentage, by 200, the base, since 12 ÷ 200 = .06 = 6%. Hence,

Rule.—*When the percentage and base are given, to find the rate, divide the percentage by the base, and the result will be the rate, expressed decimally.*

Formula, *rate* = *percentage* ÷ *base.*

Example.—Bought 300 bushels of apples and sold 228 bushels. What per cent. of the total number of bushels was sold?

Solution.—Here 300 is the base and 228 is the percentage; hence, applying rule, rate = 228 ÷ 300 = .76 = 76%. Ans.

Example.—What per cent. of 875 is 25?

Solution.—Here 875 is the base and 25 is the percentage; hence, applying rule, 25 ÷ 875 = .02$\frac{6}{7}$ = 2$\frac{6}{7}$%. Ans.

Proof.—875 × .02$\frac{6}{7}$ = 25.

EXAMPLES FOR PRACTICE.

194. What per cent. of

(*a*)	360 is 90?	Ans.	(*a*)	25%.
(*b*)	900 is 360?		(*b*)	40%.
(*c*)	125 is 25?		(*c*)	20%.
(*d*)	150 is 750?		(*d*)	500%.
(*e*)	280 is 112?		(*e*)	40%.
(*f*)	400 is 200?		(*f*)	50%.
(*g*)	47 is 94?		(*g*)	200%.
(*h*)	500 is 250?		(*h*)	50%.

195. The amount may be found, when the base and rate are given, by multiplying the base by 1 plus the rate, expressed decimally. For, suppose that it is desired to find the amount when 200 is the base and 6% is the rate. The percentage is $200 \times .06 = 12$, and, according to definition, Art. **189**, the amount is $200 + 12 = 212$. But the same result may be obtained by multiplying 200 by $1 + .06$, or 1.06, since $200 \times 1.06 = 212$. Hence,

Rule.—*When the base and rate are given, to find the amount, multiply the base by 1 plus the rate, expressed decimally.*

Formula, $amount = base \times (1 + rate)$.

EXAMPLE.—If a man earned $725 in a year, and the next year 10% more, how much did he earn the second year?

SOLUTION.—Here 725 is the base and 10% is the rate, and the amount is required. Hence, applying the rule,

$725 \times 1.10 = \$797.50$. Ans.

196. When the base and rate are given, the difference may be found by multiplying the base by 1 minus the rate, expressed decimally. For, suppose that it is desired to find the difference when the base is 200 and the rate is 6%. The percentage is $200 \times .06 = 12$; and, according to definition, Art. **190**, the difference $= 200 - 12 = 188$. But the same result may be obtained by multiplying 200 by $1 - .06$, or .94, since $200 \times .94 = 188$. Hence,

Rule.—*When the base and rate are given, to find the difference, multiply the base by 1 minus the rate, expressed decimally.*

Formula, $difference = base \times (1 - rate)$.

Example.—Bought 300 bushels of apples, and sold all but 24% of them. How many bushels were sold?

Solution.—Here 300 is the base, 24% is the rate, and it is desired to find the difference. Hence, applying the rule,

$$300 \times (1 - .24) = 228 \text{ bushels. Ans.}$$

197. When the amount and rate are given, the base may be found by dividing the amount by 1 plus the rate. For, suppose that it is known that 212 equals some number increased by 6% of itself. Then it is evident that 212 equals 106% of the number (base) that it is desired to find. Consequently, if $212 = 106\%$, $1\% = \frac{212}{106} = 2$, and $100\% = 2 \times 100 = 200 =$ the base. But the same result may be obtained by dividing 212 by $1 + .06$, or 1.06, since $212 \div 1.06 = 200$. Hence,

Rule.—*When the amount and rate are given, to find the base, divide the amount by 1 plus the rate, expressed decimally.*

Formula, *base* = *amount* ÷ (1 + *rate*).

Example.—The theoretical discharge of a certain pump, when running at a piston speed of 100 feet per minute, is 278,910 gallons per day of 10 hours. Owing to leakage and other defects, this value is 25% greater than the actual discharge. What is the actual discharge?

Solution.—Here 278,910 equals the actual discharge (base) increased by 25% of itself. Consequently, 278,910 is the amount; 25% is the rate, and, applying rule,

$$\text{actual discharge} = 278,910 \div 1.25 = 223,128 \text{ gallons. Ans.}$$

198. When the difference and rate are given, the base may be found by dividing the difference by 1 minus the rate. For, suppose that 188 equals some number less 6% of itself. Then, 188 evidently equals $100 - 6 = 94\%$ of some number. Consequently, if $188 = 94\%$, $1\% = 188 \div 94 = 2$, and $100\% = 2 \times 100 = 200$. But the same result may be obtained by dividing 188 by $1 - .06$, or .94, since $188 \div .94 = 200$. Hence,

Rule.—*When the difference and rate are given, to find the base, divide the difference by 1 minus the rate, expressed decimally.*

Formula, *base* = *difference* ÷ (1 − *rate*).

Example.—Bought a certain number of bushels of apples and sold 76% of them. If there were 72 bushels left unsold, how many bushels did I buy?

Solution.—Here 72 is the difference and 76% is the rate. Applying rule,

$$72 \div (1 - .76) = 300 \text{ bushels. Ans.}$$

Example.—The theoretical number of foot-pounds of work per minute required to operate a boiler feed-pump is 127,344. If 30% of the total number actually required be allowed for friction, leakage, etc., how many foot-pounds are actually required to work the pump?

Solution.—Here the number actually required is the base; hence, 127,344 is the difference, and 30% is the rate. Applying the rule,

$$127{,}344 \div (1 - .30) = 181{,}920 \text{ foot-pounds. Ans.}$$

199. Example.—A certain chimney gives a draft of 2.76 inches of water. By increasing the height 20 feet, the draft was increased to 3 inches of water. What was the gain per cent.?

Solution.—Here it is evident that 3 inches is the amount and that 2.76 inches is the base. Consequently, $3 - 2.76 = .24$ inch is the percentage, and it is required to find the rate. Hence, applying the rule given in Art. **193**,

$$\text{gain per cent.} = .24 \div 2.76 = .087 = 8.7\%. \text{ Ans.}$$

200. Example.—A certain chimney gave a draft of 3 inches of water. After an economizer had been put in, the draft was reduced to 1.2 inches of water. What was the loss per cent.?

Solution.—Here it is evident that 1.2 inches is the difference (since it equals 3 inches diminished by a certain per cent., loss of itself) and 3 inches is the base. Consequently, $3 - 1.2 = 1.8$ inches is the percentage. Hence, applying the rule given in Art. **193**,

$$\text{loss per cent.} = 1.8 \div 3 = .60 = 60\%. \text{ Ans.}$$

201. To find the gain or loss per cent.:

Rule.—*Find the difference between the initial and final values; divide this difference by the initial value.*

Example.—If a man buys a house for $1,860, and some time afterwards builds a barn for 25% of the cost of the house, does he gain or lose, and how much per cent., if he sells both house and barn for $2,100?

Solution.—The cost of the barn was $\$1{,}860 \times .25 = \$465$; consequently, the initial value, or cost, was $\$1{,}860 + \$465 = \$2{,}325$. Since he sold them for $2,100, he lost $\$2{,}325 - \$2{,}100 = \$225$. Hence, applying rule,

$$225 \div 2{,}325 = .0968 = 9.68\% \text{ loss. Ans.}$$

EXAMPLES FOR PRACTICE.

202. Solve the following:

(*a*)	What is $12\frac{1}{2}\%$ of \$900?	Ans.	(*a*)	\$112.50.
(*b*)	What is $\frac{4}{5}\%$ of 627?		(*b*)	5.016.
(*c*)	What is $33\frac{1}{3}\%$ of 54?		(*c*)	18.
(*d*)	101 is $68\frac{3}{4}\%$ of what number?		(*d*)	$146\frac{10}{11}$.
(*e*)	784 is $83\frac{1}{3}\%$ of what number?		(*e*)	940.8.
(*f*)	What % of 960 is 160?		(*f*)	$16\frac{2}{3}\%$.
(*g*)	What % of \$3,606 is \$450$\frac{3}{4}$?		(*g*)	$12\frac{1}{2}\%$.
(*h*)	What % of 280 is 112?		(*h*)	40%.

1. A steam plant consumed an average of 3,640 pounds of coal per day. The engineer made certain alterations which resulted in a saving of 250 pounds per day. What was the per cent. of coal saved? Ans. 7%, nearly.

2. If the speed of an engine running at 126 revolutions per minute should be increased $6\frac{1}{2}\%$, how many revolutions per minute would it then make? Ans. 134.19 revolutions.

3. The list price of a lot of silk goods is \$1,400; of some laces, \$1,150, and of some calico, \$340. If 25% discount was allowed on silk, 22% on the laces, and $12\frac{1}{2}\%$ on the calico, what was the actual cost of the purchase? Ans. \$2,244.50.

4. If I lend a man \$1,100, and this is $18\frac{1}{2}\%$ of the amount that I have on interest, how much money have I on interest? Ans. \$5,945.95.

5. A test showed that an engine developed 190.4 horsepower, 15% of which was consumed in friction. How much power was available for use? Ans. 161.84 H.P.

6. By adding a condenser to a steam engine, the power was increased 14%, and the consumption of coal per horsepower per hour was decreased 20%. If the engine could originally develop 50 horsepower, and required $3\frac{1}{2}$ pounds of coal per horsepower per hour, what would be the total weight of coal used in an hour, with the condenser, assuming the engine to run full power? Ans. 159.6 pounds.

DENOMINATE NUMBERS.

203. A **denominate number** is a concrete number, and may be either simple or compound, as 8 quarts, 5 feet, 10 inches, etc.

204. A **simple denominate number** consists of units of but one denomination, as 16 cents, 10 hours, 5 dollars, etc.

205. A **compound denominate number** consists of units of two or more denominations of a similar kind, as 3 yards 2 feet 1 inch ; 34 square feet 57 square inches.

206. In **whole numbers** and in **decimals** the *law* of increase and decrease is on the scale of 10, but in **compound** or **denominate numbers** the scale varies.

MEASURES.

207. A **measure** is a *standard unit*, established by *law* or *custom*, by which *quantity* of any kind is measured. The *standard unit* of **dry measure** is the Winchester bushel ; of **weight,** the pound ; of **liquid measure,** the gallon, etc.

208. Measures are of six kinds :

1. Extension.
2. Weight.
3. Capacity.
4. Time.
5. Angles.
6. Money or value.

MEASURES OF EXTENSION.

209. **Measures of extension** are used in measuring lengths, distances, surfaces, and solids.

LINEAR MEASURE.

TABLE 3.

			Abbreviation.	in.	ft.	yd.	rd.	fur.	mi.
12	inches (in.)	= 1 foot . .	ft.						
3	feet	= 1 yard . .	yd.	36 =	3 =	1			
5.5	yards	= 1 rod . . .	rd.	198 =	16½ =	5.5 =	1		
40	rods	= 1 furlong	fur.	7,920 =	660 =	220 =	40 =	1	
8	furlongs	= 1 mile . .	mi.	63,360 =	5,280 =	1,760 =	320 =	8 =	1

SURVEYOR'S LINEAR MEASURE.

TABLE 4.

7.92 inches = 1 link . . . . . li.
25 links = 1 rod . . . . . rd.
4 rods, 100 links = 1 chain . . . . . ch.
80 chains = 1 mile . . . . . mi.

mi.		ch.		rd.		li.		in.
1	=	80	=	320	=	8,000	=	63,360

210. The linear unit, generally used by surveyors, is **Gunter's chain,** which is equal to 4 rods, or 66 feet.

211. An **engineer's chain,** used by civil engineers, is 100 feet long, and consists of 100 links. In computations, the links are written as so many hundredths of a chain.

SQUARE MEASURE.

TABLE 5.

144 square inches (sq. in.). . . . = 1 square foot . . . . . sq.ft.
9 square feet . . . . = 1 square yard . . . . . sq.yd.
30¼ square yards . . . . = 1 square rod . . . . . sq.rd.
160 square rods . . . . = 1 acre . . . . A.
640 acres. . . . . = 1 square mile . . . . sq.mi.

sq.mi.		A.		sq.rd.		sq.yd.		sq.ft.		sq.in.
1	=	640	=	102,400	=	3,097,600	=	27,878,400	=	4,014,489,600

SURVEYOR'S SQUARE MEASURE.

TABLE 6.

625 square links . . . . = 1 square rod . . . . . sq.rd.
16 square rods. . . . . = 1 square chain . . . . sq.ch.
10 square chains . . . . = 1 acre . . . . A.
640 acres . . . . = 1 square mile. . . . . sq.mi.
36 square miles (6 mi. square) . = 1 township . . . . . . Tp.

sq.mi.		A.		sq.ch.		sq.rd.		sq.li.
1	=	640	=	6,400	=	102,400	=	64,000,000

CUBIC MEASURE.

TABLE 7.

1728 cubic inches (cu. in.) . . . . = 1 cubic foot . . . . . cu.ft.
27 cubic feet . . . . = 1 cubic yard . . . . . cu.yd.
128 cubic feet . . . . = 1 cord . . . . cd.
24¾ cubic feet . . . . = 1 perch . . . . P.

cu.yd.		cu.ft.		cu.in.
1	=	27	=	46,656

MEASURES OF WEIGHT.

AVOIRDUPOIS WEIGHT.

TABLE 8.

16 ounces (oz.). . . . . = 1 pound . . . . lb.
100 pounds . . . . = 1 hundredweight . . . . cwt.
20 cwt., or 2,000 lb. . . . . . . = 1 ton . . . . T.

T.		cwt.		lb.		oz.
1	=	20	=	2,000	=	32,000

212. The ounce is divided into halves, quarters, etc. Avoirdupois weight is used for weighing coarse and heavy articles. One avoirdupois pound contains 7,000 grains.

LONG TON TABLE.

TABLE 9.

16 ounces . . . . = 1 pound . . . . lb.
112 pounds . . . . = 1 hundredweight . . . cwt.
20 cwt., or 2,240 lb. . . . . . . = 1 ton . . . . T.

213. In all the calculations throughout this and the succeeding volumes, 2,000 pounds will be considered one ton, unless the long ton (2,240 pounds) is especially mentioned.

TROY WEIGHT.

TABLE 10.

24 grains (gr.) . . . . = 1 pennyweight . . . . pwt.
20 pennyweights . . . . = 1 ounce . . . . oz.
12 ounces . . . . = 1 pound . . . . lb.

lb. oz. pwt. gr.
1 = 12 = 240 = 5,760

214. Troy weight is used in weighing gold and silverware, jewels, etc. It is used by jewelers.

MEASURES OF CAPACITY.

LIQUID MEASURE.

TABLE 11.

4 gills (gi.). . . . . = 1 pint . . . . pt.
2 pints . . . . = 1 quart. . . . . qt.
4 quarts . . . . = 1 gallon . . . . gal.
31½ gallons . . . . = 1 barrel . . . . bbl.
2 barrels, or 63 gallons . . . . = 1 hogshead . . . . hhd.

hhd. bbl. gal. qt. pt. gi.
1 = 2 = 63 = 252 = 504 = 2,016

DRY MEASURE.

TABLE 12.

2 pints (pt.) . . . . = 1 quart . . . . qt.
8 quarts . . . . = 1 peck . . . . pk.
4 pecks. . . . . = 1 bushel. . . . . bu.

bu. pk. qt. pt.
1 = 4 = 32 = 64

MEASURE OF TIME.

TABLE 13.

60 seconds (sec.)	=	1 minute	min.
60 minutes	=	1 hour	hr.
24 hours	=	1 day	da.
7 days	=	1 week	wk.
365 days 12 months	=	1 common year	yr.
366 days	=	1 leap year.	
100 years	=	1 century.	

NOTE.—It is customary to consider one month as 30 days.

MEASURE OF ANGLES OR ARCS.

TABLE 14.

60 seconds (″)	=	1 minute	′.
60 minutes	=	1 degree	°.
90 degrees	=	1 right angle or quadrant	∟.
360 degrees	=	1 circle	cir.

cir.
1 = 360° = 21,600′ = 1,296,000″

MEASURE OF MONEY.

UNITED STATES MONEY.

TABLE 15.

10 mills (m.)	=	1 cent	ct.
10 cents	=	1 dime	d.
10 dimes	=	1 dollar	$.
10 dollars	=	1 eagle	E.

E.		$		d.		ct.		m.
1	=	10	=	100	=	1,000	=	10,000

MISCELLANEOUS TABLE.

TABLE 16.

12 things are 1 dozen.
12 dozen are 1 gross.
12 gross are 1 great gross.
2 things are 1 pair.
20 things are 1 score.
1 league is 3 miles.
1 fathom is 6 feet.

1 meter is nearly 39.37 inches.
1 hand is 4 inches.
1 palm is 3 inches.
1 span is 9 inches.
24 sheets are 1 quire.
20 quires, or 480 sheets, are 1 ream.
1 bushel contains 2,150.4 cubic in.

1 U. S. standard gallon (also called a wine gallon) contains 231 cubic in.
1 U. S. standard gallon of water weighs 8.355 pounds, nearly.
1 cubic foot of water contains 7.481 U. S. standard gallons, nearly.
1 British imperial gallon weighs 10 pounds.

It will be of great advantage to the student to carefully memorize all of the above tables.

REDUCTION OF DENOMINATE NUMBERS.

215. Reduction of denominate numbers is the process of changing their denomination without changing their value. They may be changed from a higher to a lower denomination or from a lower to a higher—either is reduction. As,

2 hours = 120 minutes.
32 ounces = 2 pounds.

216. Principle.—Denominate numbers are changed to *lower* denominations by *multiplying*, and to *higher* denominations by *dividing*.

To reduce denominate numbers to lower denominations:

217. EXAMPLE.—Reduce 5 yd. 2 ft. 7 in. to inches.

SOLUTION.—

$$\begin{array}{rlll}
 & \text{yd.} & \text{ft.} & \text{in.} \\
 & 5 & 2 & 7 \\
 & 3 & & \\
\hline
 & 15 \text{ ft.} & & \\
 & 2 \text{ ft.} & & \\
\hline
 & 17 \text{ ft.} & & \\
 & 12 & & \\
\hline
 & 34 & & \\
 & 17 & & \\
\hline
 & 204 \text{ in.} & & \\
 & 7 \text{ in.} & & \\
\hline
 & 211 \text{ inches. Ans.} & &
\end{array}$$

EXPLANATION.—Since there are 3 feet in 1 yard, in 5 yards there are 5 × 3, or 15 feet, and 15 feet plus 2 feet = 17 feet. There are 12 inches in a foot; therefore, 12 × 17 = 204 inches, and 204 inches plus 7 inches = 211 inches = number of inches in 5 yards 2 feet and 7 inches. Ans.

218. EXAMPLE.—Reduce 6 hours to seconds.

SOLUTION.—

$$\begin{array}{rl}
6 & \text{hours.} \\
60 & \\
\hline
360 & \text{minutes.} \\
60 & \\
\hline
21600 & \text{seconds. Ans.}
\end{array}$$

EXPLANATION.—As there are 60 minutes in one hour, in six hours there are 6 × 60, or 360 minutes; as there are no minutes to add, we multiply 360 minutes by 60, to get the number of seconds.

219. In order to avoid mistakes, if any denomination be omitted, represent it by a cipher. Thus, before reducing 3 rods 6 inches to inches, insert a cipher for yards and a cipher for feet; as,

rd.	yd.	ft.	in.
3	0	0	6

220. **Rule.**—*Multiply the number representing the highest denomination by the number of units in the next lower required to make one of the higher denomination, and to the product add the number of given units of that lower denomination. Proceed in this manner until the number is reduced to the required denomination.*

EXAMPLES FOR PRACTICE.

221. Reduce

			Ans.
(*a*)	4 rd. 2 yd. 2 ft. to ft.	(*a*)	74 ft.
(*b*)	4 bu. 3 pk. 2 qt. to qt.	(*b*)	154 qt.
(*c*)	13 rd. 5 yd. 2 ft. to ft.	(*c*)	231.5 ft.
(*d*)	5 mi. 100 rd. 10 ft. to ft.	(*d*)	28,060 ft.
(*e*)	8 lb. 4 oz. 6 pwt. to gr.	(*e*)	48,144 gr.
(*f*)	52 hhd. 24 gal. 1 pt. to pt.	(*f*)	26,401 pt.
(*g*)	5 cir. 16° 20′ to minutes.	(*g*)	108,980′.
(*h*)	14 bu. to qt.	(*h*)	448 qt.

To reduce lower to higher denominations:

222. EXAMPLE.—Reduce 211 in. to higher denominations.

SOLUTION.—
12) 211 in.
3) 17 ft. + 7 in.
5 yd. + 2 ft. Ans.

EXPLANATION.—There are 12 inches in 1 foot; therefore, 211 divided by 12 = 17 feet and 7 inches over. There are 3 feet in 1 yard; therefore, 17 feet divided by 3 = 5 yards and 2 feet over. The last quotient and the two remainders constitute the answer, 5 yards 2 feet 7 inches.

223. Example.—Reduce 15,735 grains Troy weight to higher denominations.

Solution.—

24)15735 gr.(655 pwt.
144
133
120
135
120
15 gr.

20)655 pwt.(32 oz.
60
55
40
15 pwt.

12)32 oz.(2 lb.
24
8 oz.

Explanation.—There are 24 grains in 1 pennyweight, and in 15,735 grains there are as many pennyweights as 24 is contained in 15,735, or 655 pennyweights and 15 grains remaining. There are 20 pennyweights in 1 ounce, and in 655 pennyweights there are 32 ounces and 15 pennyweights remaining. There are 12 ounces in 1 pound, and in 32 ounces there are 2 pounds and 8 ounces remaining. The last quotient and the three remainders constitute the answer, 2 pounds 8 ounces 15 pennyweights 15 grains.

The above problem is worked out by long division, because the numbers are too large to solve easily by short division. The student may use either method.

224. Rule.—*Divide the number representing the denomination given by the number of units of this denomination required to make one unit of the next higher denomination. The remainder will be of the same denomination, but the quotient will be of the next higher. Divide this quotient by* *the number of units of its denomination required to make one unit of the next higher. Continue until the highest denomination is reached, or until there is not enough of a denomination left to make one of the next higher. The last quotient and the remainders constitute the required result.*

EXAMPLES FOR PRACTICE.

225. Reduce to units of higher denominations:

(*a*) 7,460 sq. in.; (*b*) 7,580 sq. yd.; (*c*) 148,760 cu. in.; (*d*) 7,896 cu. ft. to cd.; (*e*) 17,651″; (*f*) 1,120 cu. ft. to cd.; (*g*) 8,000 gi.; (*h*) 36,450 lb.

Ans.
(*a*) 5 sq. yd. 6 sq. ft. 116 sq. in.
(*b*) 1 A. 90 sq. rd. 17 sq. yd. 4 sq. ft. 72 sq. in
(*c*) 3 cu. yd. 5 cu. ft. 152 cu. in.
(*d*) 61 cd. 88 cu. ft.
(*e*) 4° 54′ 11″.
(*f*) 8 cd. 96 cu. ft.
(*g*) 3 hhd. 61 gal.
(*h*) 18 T. 4 cwt. 50 lb.

ADDITION OF DENOMINATE NUMBERS.

226. Example.—Find the sum of 3 cwt. 46 lb. 12 oz.; 8 cwt. 12 lb 13 oz.; 12 cwt. 50 lb. 13 oz.; 27 lb. 4 oz.

Solution.—

T.	cwt.	lb.	oz.	
0	3	46	12	
0	8	12	13	
0	12	50	13	
0	0	27	4	
1	4	37	10	Ans.

Explanation.—Begin to add at the right-hand column: $4 + 13 + 13 + 12 = 42$ ounces: as 16 ounces make 1 pound, 42 ounces $\div 16 = 2$ and a remainder of 10 ounces, or 2 pounds and 10 ounces. Place 10 ounces under ounce column, and add 2 pounds to the next or pound column. Then, $2 + 27 + 50 + 12 + 46 = 137$ pounds; as 100 pounds make a hundredweight, $137 \div 100 = 1$ hundredweight and a remainder of 37 pounds. Place the 37 under the pounds column, and add 1 hundredweight to the next or hundredweight column. Next, $1 + 12 + 8 + 3 = 24$ hundredweight.

20 hundredweight make a ton; therefore 24 ÷ 20 = 1 ton and 4 hundredweight remaining. Hence, the sum is 1 ton 4 hundredweight 37 pounds 10 ounces. Ans.

227. EXAMPLE.—What is the sum of 2 rd. 3 yd. 2 ft. 5 in.; 6 rd. 1 ft. 10 in.; 17 rd. 11 in.; 4 yd. 1 ft.?

SOLUTION.—

	rd.	yd.	ft.	in.	
	2	3	2	5	
	6	0	1	10	
	17	0	0	11	
	0	4	1	0	
	26	3½	0	2	
or	26	3	1	8	Ans.

EXPLANATION.—The sum of the numbers in the first column = 26 inches, or 2 feet and 2 inches remaining. The sum of the numbers in the next column plus 2 feet = 6 feet, or 2 yards and 0 feet remaining. The sum of the next column plus 2 yards = 9 yards, or 9 ÷ 5½ = 1 rod and 3½ yards remaining. The sum of the next column plus 1 rod = 26 rods. To avoid fractions in the sum, the ½ yard is reduced to 1 foot and 6 inches, which added to 26 rods 3 yards 0 feet and 2 inches = 26 rods 3 yards 1 foot 8 inches. Ans.

228. EXAMPLE.—What is the sum of 47 ft. and 3 rd. 2 yd. 2 ft. 10 in.?

SOLUTION.—When 47 ft. is reduced it equals 2 rd. 4 yd. 2 ft., which can be added to 3 rd. 2 yd. 2 ft. 10 in. Thus,

	rd.	yd.	ft.	in.	
	3	2	2	10	
	2	4	2	0	
	6	1½	1	10	
or	6	2	0	4	Ans.

229. Rule.—*Place the numbers so that like denominations are under each other. Begin at the right-hand column, and add. Divide the sum by the number of units of this denomination required to make one unit of the next higher. Place the remainder under the column added, and carry the quotient to the next column. Continue in this manner until the highest denomination given is reached.*

EXAMPLES FOR PRACTICE.

230. What is the sum of

(*a*) 25 lb. 7 oz. 15 pwt. 23 gr.; 17 lb. 16 pwt.; 15 lb. 4 oz. 12 pwt.; 18 lb. 16 gr.; 10 lb. 2 oz. 11 pwt. 16 gr.?

(*b*) 9 mi. 13 rd. 4 yd. 2 ft.; 16 rd. 5 yd. 1 ft. 5 in.; 16 mi. 2 rd. 8 in.; 14 rd. 1 yd. 9 in.?

(*c*) 3 cwt. 46 lb. 12 oz.; 12 cwt. 9¼ lb.; 2½ cwt. 21⅜ lb.?

(*d*) 10 yr. 8 mo. 5 wk. 3 da.; 42 yr. 6 mo. 7 da.; 7 yr. 5 mo. 18 wk. 4 da.; 17 yr. 17 da.?

(*e*) 17 tons 11 cwt. 49 lb. 14 oz.; 16 tons 47 lb. 13 oz.; 20 tons 13 cwt. 14 lb. 6 oz.; 11 tons 4 cwt. 16 lb. 12 oz.?

(*f*) 14 sq. yd. 8 sq. ft. 19 sq. in.; 105 sq. yd. 16 sq. ft. 240 sq. in., 42 sq. yd. 28 sq. ft. 165 sq. in.?

Ans. {
(*a*) 86 lb. 3 oz. 16 pwt. 7 gr.
(*b*) 25 mi. 47 rd. 1 ft. 5 in.
(*c*) 18 cwt. 2 lb. 14 oz.
(*d*) 78 yr. 1 mo. 3 wk. 3 da.
(*e*) 65 tons 9 cwt. 28 lb. 13 oz.
(*f*) 167 sq. yd. 136 sq. in.

SUBTRACTION OF DENOMINATE NUMBERS.

231. Example.—From 21 rd. 2 yd. 2 ft. 6½ in., take 9 rd 4 yd. 10¼ in.

Solution.—

rd.	yd.	ft.	in.	
21	2	2	6½	
9	4	0	10¼	
11	3½	1	8¼	Ans.

Explanation.—Since 10¼ inches cannot be taken from 6½ inches, we must borrow 1 foot, or 12 inches, from the 2 feet in the next column and add it to the 6½. 6½ + 12 = 18½. 18½ inches − 10¼ inches = 8¼ inches. Then, 0 foot from the 1 remaining foot = 1 foot. 4 yards cannot be taken from 2 yards; therefore, we borrow 1 rod, or 5½ yards, from 21 rods and add it to 2. 2 + 5½ = 7½; 7½ − 4 = 3½ yards. 9 rods from 20 rods = 11 rods. Hence, the remainder is 11 rods 3½ yards 1 foot 8¼ inches. Ans.

To avoid fractions as much as possible, we reduce the ½ yard to inches, obtaining 18 inches; this added to 8¼ inches, gives 26¼ inches, which equals 2 feet 2¼ inches. Then, 2 feet + 1 foot = 3 feet = 1 yard, and 3 yards + 1 yard = 4 yards. Hence, the above answer becomes 11 rods 4 yards 0 feet 2¼ inches.

232. EXAMPLE.—What is the difference between 3 rd. 2 yd. 2 ft. 10 in. and 47 ft.?

SOLUTION.—47 ft. = 2 rd. 4 yd. 2 ft.

	rd.	yd.	ft.	in.	
	3	2	2	10	
	2	4	2	0	
	0	3½	0	10	
or		3	2	4	Ans.

To find (approximately) the interval of time between two dates:

233. EXAMPLE.—How many years, months, days, and hours between 4 o'clock P.M. of June 15, 1868, and 10 o'clock A.M., September 28, 1891?

SOLUTION.—

yr.	mo.	da.	hr.	
1891	8	28	10	
1868	5	15	16	
23	3	12	18	Ans.

EXPLANATION.—Counting 24 hours in 1 day, 4 o'clock P.M. is the 16th hour from the beginning of the day, or midnight. On September 28, 8 months and 28 days have elapsed, and on June 15, 5 months and 15 days. After placing the earlier date under the later date, subtract as in the previous problems. Count 30 days as 1 month.

234. Rule.—*Place the smaller quantity under the larger quantity, with like denominations under each other. Beginning at the right, subtract successively the number in the subtrahend in each denomination from the one above, and place the differences underneath. If the number in the minuend of any denomination is less than the number under it in the subtrahend, one must be borrowed from the minuend of the next higher denomination, reduced and added to it.*

EXAMPLES FOR PRACTICE.

235. From

(*a*) 125 lb. 8 oz. 14 pwt. 18 gr. take 96 lb. 9 oz. 10 pwt. 4 gr.
(*b*) 126 hhd. 27 gal. take 104 hhd. 14 gal. 1 qt. 1 pt.
(*c*) 65 T. 14 cwt. 64 lb. 10 oz. take 16 T. 11 cwt. 14 oz.
(*d*) 148 sq. yd. 16 sq. ft. 142 sq. in. take 132 sq. yd. 136 sq. in.

(*e*) 100 bu. take 28 bu. 2 pk. 5 qt. 1 pt.
(*f*) 14 mi. 34 rd. 16 yd. 13 ft. 11 in. take 3 mi. 27 rd. 11 yd. 4 ft. 10 in.

Ans.
(*a*) 28 lb. 11 oz. 4 pwt. 14 gr.
(*b*) 22 hhd. 12 gal. 2 qt. 1 pt.
(*c*) 49 T. 3 cwt. 63 lb. 12 oz.
(*d*) 16 sq. yd. 16 sq. ft. 6 sq. in.
(*e*) 71 bu. 1 pk. 2 qt. 1 pt.
(*f*) 11 mi. 7 rd. 5 yd. 9 ft. 1 in.

MULTIPLICATION OF DENOMINATE NUMBERS.

236. Example.—Multiply 7 lb. 5 oz. 13 pwt. 15 gr. by 12.

Solution.—

lb.	oz.	pwt.	gr.	
7	5	13	15	
			12	
89	8	3	12	Ans.

Explanation.—15 grains × 12 = 180 grains. 180 ÷ 24 = 7 pennyweights and 12 grains remaining. Place the 12 in the grain column and carry the 7 pennyweights to the next. Now, 13 × 12 + 7 = 163 pennyweights ; 163 ÷ 20 = 8 ounces and 3 pennyweights remaining. Then, 5 × 12 + 8 = 68 ounces; 68 ÷ 12 = 5 pounds and 8 ounces remaining. Then, 7 × 12 + 5 = 89 pounds. The entire product is 89 pounds 8 ounces 3 pennyweights 12 grains. Ans.

237. Rule.—*Multiply the number representing each denomination by the multiplier, and reduce each product to the next higher denomination, writing the remainders under each denomination, and carrying the quotient to the next, as in Addition of Denominate Numbers.*

238. Note.—In multiplication and division of denominate numbers, it is sometimes easier to reduce the number to the lowest denomination given before multiplying or dividing, especially if the multiplier or divisor is a decimal. Thus, in the above example, had the multiplier been 1.2, the easiest way to multiply would have been to reduce the number to grains; then, multiply by 1.2, and reduce the product to higher denominations. For example, 7 lb. 5 oz. 13 pwt. 15 gr. = 43,047 gr. 43,047 × 1.2 = 51,656.4 gr. = 8 lb. 11 oz. 12 pwt. 8.4 gr. Also, 43,047 × 12 = 516,564 gr. = 89 lb. 8 oz. 3 pwt. 12 gr., as above. The student may use either method.

EXAMPLES FOR PRACTICE.

239. Multiply

(*a*) 15 cwt. 90 lb. by 5; (*b*) 12 yr. 10 mo. 4 wk. 3 da. by 14; (*c*) 11 mi. 145 rd. by 20; (*d*) 12 gal. 4 pt. by 9; (*e*) 8 cd. 76 cu. ft. by 15; (*f*) 4 hhd. 8 gal. 1 qt. 1 pt. by 12.

Ans.
- (*a*) 79 cwt. 50 lb.
- (*b*) 180 yr. 11 mo. 2 wk.
- (*c*) 229 mi. 20 rd.
- (*d*) 112 gal. 2 qt.
- (*e*) 128 cd. 116 cu. ft.
- (*f*) 48 hhd. 40 gal. 2 qt.

DIVISION OF DENOMINATE NUMBERS.

240. Example.—Divide 48 lb. 11 oz. 6 pwt. by 8.

Solution.—

	lb.	oz.	pwt.	gr.	
8)	48	11	6	0	
	6 lb.	1 oz.	8 pwt.	6 gr.	Ans.

Explanation.—After placing the quantities as above, proceed as follows: 8 is contained in 48 six times without a remainder. 8 is contained in 11 ounces once with 3 ounces remaining. 3 × 20 = 60; 60 + 6 = 66 pennyweights; 66 pennyweights ÷ 8 = 8 pennyweights and 2 remaining; 2 × 24 grains = 48 grains; 48 grains ÷ 8 = 6 grains. Therefore, the entire quotient is 6 pounds 1 ounce 8 pennyweights 6 grains. Ans.

Example.—A silversmith melted up 2 lb. 8 oz. 10 pwt. of silver, which he made into 6 spoons; what was the weight of each spoon?

Solution.—

	lb.	oz.	pwt.		
6)	2	8	10		
		5 oz.	8 pwt.	8 gr.	Ans.

Explanation.—Since we cannot divide 2 pounds by 6, we reduce it to ounces. 2 pounds = 24 ounces, and 24 ounces + 8 ounces = 32 ounces; 32 ounces ÷ 6 = 5 ounces and 2 ounces over. 2 ounces = 40 pennyweights. 40 pennyweights + 10 pennyweights = 50 pennyweights, and 50 pennyweights ÷ 6 = 8 pennyweights and 2 pennyweights over. 2 pennyweights = 48 grains, and 48 grains ÷ 6 = 8 grains. Hence, each spoon contains 5 ounces 8 pennyweights 8 grains. Ans.

241. Example.—Divide 820 rd. 4 yd. 2 ft. by 112.

Solution.—

```
           rd.   yd. ft. rd. yd. ft.   in.
       112)820    4   2 ( 7   1   2  5.143  Ans.
           784
           ---
            36 rd. rem.
            5.5
           ---
           180
          180
          -----
          198.0 yd.
            4
          ---
      112)202 yd. (1 yd.
          112
          ---
           90 yd. rem.
            3
          ---
          270 ft.
            2 ft.
          ---
      112)272 ft. (2 ft.
          224
          ---
           48 ft. rem.
           12
           --
           96
          48
          ---
      112)576 in. (5.1428 + in., or 5.143 in.
          560
          ---
           160
           112
           ---
            480
            448
            ---
             320
             224
             ---
              960
              896
              ---
               64
```

Explanation.—The first quotient is 7 rods with 36 rods remaining. 5.5 × 36 = 198 yards; 198 yards + 4 yards = 202 yards; 202 yards ÷ 112 = 1 yard and 90 yards remaining. 90 × 3 = 270 feet; 270 feet + 2 feet = 272 feet; 272 feet ÷ 112 = 2 feet and 48 feet remaining; 48 × 12 = 576 inches; 576 inches ÷ 112 = 5.143 inches, nearly. Ans.

The preceding example is solved by long division, because the numbers are too large to deal with mentally. Instead of expressing the last result as a decimal, it might have been expressed as a common fraction. Thus, 576 ÷ 112 = $5\frac{16}{112} = 5\frac{1}{7}$ inches. The chief advantage of using a common fraction is that if the quotient be multiplied by the divisor, the result will always be the same as the original dividend.

242. Rule.—*Find how many times the divisor is contained in the first or highest denomination of the dividend. Reduce the remainder (if any) to the next lower denomination, and add to it the number in the given dividend expressing that denomination. Divide this new dividend by the divisor. The quotient will be the next denomination in the quotient required. Continue in this manner until the lowest denomination is reached. The successive quotients will constitute the entire quotient.*

EXAMPLES FOR PRACTICE.

243. Divide

(*a*) 376 mi. 276 rd. by 22; (*b*) 1,187 bu. 3 pk. 4 qt. 1 pt. by 10; (*c*) 84 cwt. 48 lb. 49 oz. by 16; (*d*) 78 sq. yd. 18 sq. ft. 41 sq. in. by 18; (*e*) 148 mi. 64 rd. 24 yd. by 12; (*f*) 100 tons 16 cwt. 18 lb. 11 oz. by 15; (*g*) 36 lb. 18 oz. 18 pwt. 14 gr. by 8; (*h*) 112 mi. 48 rd. by 100.

Ans.
(*a*) 17 mi. $41\frac{7}{11}$ rd.
(*b*) 118 bu. 3 pk. 1 qt. $\frac{1}{2}$ pt.
(*c*) 5 cwt. 28 lb. $3\frac{1}{16}$ oz.
(*d*) 4 sq. yd. 4 sq. ft. $2\frac{5}{18}$ sq. in.
(*e*) 12 mi. 112 rd. 2 yd.
(*f*) 6 tons 14 cwt. 41 lb. $3\frac{14}{15}$ oz.
(*g*) 4 lb. 8 oz. 7 pwt. $7\frac{3}{4}$ gr.
(*h*) 1 mi. $38\frac{22}{25}$ rd.

ARITHMETIC.

(SECTION 5.)

INVOLUTION.

244. If a product consists of equal factors, it is called a **power** of one of those equal factors, and one of the equal factors is called a **root** of the product. The power and the root are named according to the number of equal factors in the product. Thus, 3 × 3, or 9, is the *second power*, or **square,** of 3; 3 × 3 × 3, or 27, is the *third power*, or **cube,** of 3; 3 × 3 × 3 × 3, or 81, is the **fourth power** of 3. Also, 3 is the **second root,** or **square root,** of 9; 3 is the **third root,** or **cube root,** of 27; 3 is the **fourth root** of 81.

245. For the sake of brevity,

3 × 3 is written 3^2, and read **three square,**
or *three exponent two;*
3 × 3 × 3 is written 3^3, and read **three cube,**
or *three exponent three;*
3 × 3 × 3 × 3 is written 3^4, and read **three fourth,**
or *three exponent four;*

and so on.

A number written above and to the right of another number, to show how often the latter number is used as a factor, is called an **exponent.** Thus, in 3^{12}, the number ^{12} is the exponent, and shows that 3 is to be used as a factor twelve times; so that 3^{12} is a contraction for

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3.$$

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In an expression like 3^6, the exponent ^{6} shows how often 3 is used as a factor. Hence, if the exponent of a number is unity, the number is used once as a factor; thus, $3^1 = 3$, $4^1 = 4$, $5^1 = 5$.

246. If the side of a square contains 5 inches, the area of the square contains 5×5, or 5^2, square inches. If the edge of a cube contains 5 inches, the volume of the cube contains $5 \times 5 \times 5$, or 5^3, cubic inches. It is for this reason that 5^2 and 5^3 are called the square and cube of 5, respectively.

247. To find any power of a number:

EXAMPLE 1.—What is the third power, or cube, of 35?

SOLUTION.— $35 \times 35 \times 35$

$$
\begin{array}{r|ccccc}
\text{or} & & & & 3 & 5 \\
 & & & & 3 & 5 \\
\cline{2-6}
 & & & 1 & 7 & 5 \\
 & & 1 & 0 & 5 & \\
\cline{2-6}
 & & 1 & 2 & 2 & 5 \\
 & & & & 3 & 5 \\
\cline{2-6}
 & & 6 & 1 & 2 & 5 \\
 & 3 & 6 & 7 & 5 & \\
\cline{2-6}
\textit{cube} = & 4 & 2 & 8 & 7 & 5 \quad \text{Ans.}
\end{array}
$$

EXAMPLE 2.—What is the fourth power of 15?

SOLUTION.— $15 \times 15 \times 15 \times 15$

$$
\begin{array}{r|ccccc}
\text{or} & & & & 1 & 5 \\
 & & & & 1 & 5 \\
\cline{2-6}
 & & & & 7 & 5 \\
 & & & 1 & 5 & \\
\cline{2-6}
 & & & 2 & 2 & 5 \\
 & & & & 1 & 5 \\
\cline{2-6}
 & & 1 & 1 & 2 & 5 \\
 & & 2 & 2 & 5 & \\
\cline{2-6}
 & & 3 & 3 & 7 & 5 \\
 & & & & 1 & 5 \\
\cline{2-6}
 & 1 & 6 & 8 & 7 & 5 \\
 & 3 & 3 & 7 & 5 & \\
\cline{2-6}
\textit{fourth power} = & 5 & 0 & 6 & 2 & 5 \quad \text{Ans.}
\end{array}
$$

Example 3.— 1.2^3 = what?

Solution.—

$$1.2 \times 1.2 \times 1.2$$

$$\text{or}\quad \begin{array}{r} 1.2 \\ 1.2 \\ \hline 1.44 \\ 1.2 \\ \hline 288 \\ 144 \\ \hline \end{array}$$

$$\textit{cube} = 1.728 \quad \text{Ans.}$$

Example 4.—What is the third power, or cube, of $\frac{3}{8}$?

Solution.— $\left(\frac{3}{8}\right)^3 = \frac{3^3}{8^3} = \frac{3}{8} \times \frac{3}{8} \times \frac{3}{8} = \frac{3 \times 3 \times 3}{8 \times 8 \times 8} = \frac{27}{512}$. Ans.

248. Rule.—I. *To raise a whole number or a decimal to any power, use it as a factor as many times as there are units in the exponent.*

II. *To raise a fraction to any power, raise both the numerator and denominator to the power indicated by the exponent.*

EXAMPLES FOR PRACTICE.

Raise the following to the powers indicated:

(*a*)	85^2.	Ans.	(*a*)	7,225.
(*b*)	$\left(\frac{12}{13}\right)^2$.		(*b*)	$\frac{144}{169}$.
(*c*)	6.5^2.		(*c*)	42.25.
(*d*)	14^4.		(*d*)	38,416.
(*e*)	$\left(\frac{3}{4}\right)^3$.		(*e*)	$\frac{27}{64}$.
(*f*)	$\left(\frac{5}{6}\right)^3$.		(*f*)	$\frac{125}{216}$.
(*g*)	$\left(\frac{7}{8}\right)^3$.		(*g*)	$\frac{343}{512}$.
(*h*)	1.4^5.		(*h*)	5.37824.

EVOLUTION.

249. Evolution is the reverse of involution. It is the process of finding the root of a number that is considered as a power.

250. The **square root** of a number is that number which when used twice as a factor produces the number.

Thus, 2 is the square root of 4, since 2×2, or $2^2 = 4$.

251. The **cube root** of a number is that number which when used three times as a factor produces the number.

Thus, 3 is the cube root of 27, since $3 \times 3 \times 3$, or $3^3 = 27$.

252. The **radical sign** $\sqrt{}$ when placed before a number indicates that some root of that number is to be found. The vinculum is almost always used in connection with the radical sign, as shown in Art. **253.**

253. The **index** of the root is a *small figure* placed *over* and to the *left* of the *radical sign*, to show what root is to be found.

Thus, $\sqrt[2]{100}$ denotes the *square root* of 100.
$\sqrt[3]{125}$ denotes the *cube root* of 125.
$\sqrt[4]{256}$ denotes the *fourth root* of 256, and so on.

254. When the square root is to be extracted, the index is generally omitted. Thus, $\sqrt{100}$ indicates the square root of 100. Also, $\sqrt{225}$ indicates the square root of 225.

255. In any number, the figures beginning with the first digit* at the left and ending with the last digit at the right, are called the **significant figures** of the number. Thus, the number 405,800 has the four significant figures 4, 0, 5, 8; and the number .000090067 has the five significant figures 9, 0, 0, 6, and 7.

The part of a number consisting of its significant figures is called the **significant part** of the number. Thus, in the number 28,070, the significant part is 2807; in the number .00812, the significant part is 812; and in the number 170.3, the significant part is 1703.

In speaking of the significant figures or of the significant part of a number, we consider the figures, in their proper order, from the first digit at the left to the last digit at the right, but we pay no attention to the position of the decimal point. Hence, *all numbers that differ only in the position of the decimal point have the same significant part.* For example, .002103, 21.03, 21,030, and 210,300 have the same significant figures 2, 1, 0, and 3 and the same significant part 2103.

---

* A cipher is not a digit.

SQUARE ROOT.

256. The *largest* number that can be written with *one* figure is 9, and $9^2 = 81$; the *largest* number that can be written with *two* figures is 99, and $99^2 = 9{,}801$; with *three* figures 999, and $999^2 = 998{,}001$; with *four* figures 9,999, and $9{,}999^2 = 99{,}980{,}001$, etc.

In *each* of the above it will be noticed that the square of the number contains just *twice* as many figures as the number.

In order to find the square root of a number, the first step is to find how many figures there will be in the root. This is done by pointing off the number into *periods* of *two* figures each, *beginning at the right.* The number of periods will indicate the number of figures in the root.

Thus, the square root of 83,740,801 must contain 4 figures, since, pointing off the periods, we get 83'74'08'01, or 4 periods; consequently, there must be 4 figures in the root. In like manner, the square root of 50,625 must contain 3 figures, since there are (5'06'25) 3 periods. The extreme left-hand period may contain either one or two figures, according to the size of the number squared.

257. The square of any number wholly decimal always contains twice as many figures as the number squared. For example, $.1^2 = .01$, $.13^2 = .0169$, $.751^2 = .564001$, etc.

258. It will also be noticed that the square of a decimal is always less than the decimal. Hence, the square root of a number wholly decimal is greater than the number itself. If it be required to find the square root of a decimal, and the decimal has not an even number of figures in it, annex a cipher. The best way to point off a decimal is to begin at the decimal point, and, going towards the *right*, point off the decimal into periods of two figures each. Then, if the last period contains but one figure, annex a cipher to complete the period.

259. There are comparatively few numbers that can be separated into exactly equal factors; these numbers are called

perfect powers, and the factors are called *rational factors.* Numbers that cannot be separated into exactly equal factors are called **imperfect powers,** and the factors are called **surds** or *irrational factors.* In the numbers from 1 to 1,000, inclusive, there are only 42 perfect powers, not counting 1; and of these only 30 are perfect squares and 9 perfect cubes.

The root of any number that cannot be divided into as many equal factors as there are units in the index of the root contains an interminable decimal. For example, the number 20 lies between 16 ($= 4^2$) and 25 ($= 5^2$); hence, the square root of 20, or $\sqrt{20}$, is greater than 4 and less than 5, and is therefore equal to 4 plus an interminable decimal. In other words, no matter to how many figures the square root of 20 may be calculated, the root will never be found exactly.

260. Although the root of an imperfect power cannot be found exactly, as close an approximation may be obtained as is desired. In practice, five significant figures are all that are likely to be required, and four are generally sufficient. In the following examples, all roots will be calculated to five figures, unless the given number is a perfect power whose root contains less than five figures.

261. The student will find the following principles of value, both in connection with the extraction of roots and in other arithmetical calculations:

a. In general, if any two numbers are multiplied together—no matter how many significant figures they contain—the first five significant figures of the product will be the same as the first five significant figures of the product obtained by multiplying the same two numbers when limited to five significant figures.

For example, the product of 4,562,357 and 6,421,849 is 29,298,767,738,093; limiting the numbers to five significant figures, the product of 45,624 and 64,218 is 2,929,882,032; and the value of both these products to five significant figures is 29,299. In other words, if only five significant figures are required in the product, it is not necessary to use more than five significant figures in the multiplier and multiplicand, the remaining figures, if any, being replaced by ciphers, and the fifth figures being increased by 1 if the sixth figure is 5 or a larger digit. In some cases, however, the fifth figure may be one unit too large or one unit too small; hence, if it is necessary that the fifth figure be absolutely exact, it is better to limit the multiplier and multiplicand to six figures instead of five.

For example, 4,562,347 × 6,421,849 = 29,298,703,519,603, or 29,299,000,000,000 to five significant figures; 4,562,300 × 6,421,800 = 29,298,178,140,000 = 29,298,000,000,000 to five significant figures, the fifth figure being 1 less than it should be; but 4,562,350 × 6,421,850 = 29,298,727,347,500 = 29,299,000,000,000 to five significant figures.

b. If the divisor and dividend are limited to six significant figures, the quotient will always be correct to five (usually to six) significant figures, regardless of how many significant figures there may have been in the dividend and divisor.

For example, 6,421,849 ÷ 4,562,357 = 1.407572+ = 1.4076 to five significant figures; also, 642,185 ÷ 456,236 = 1.407571+ = 1.4076 to five significant figures.

c. If the number whose root is to be extracted be limited to six significant figures, the root will be correct to five (usually to six) significant figures.

262. These principles may all be summed up in the following general statement: *In any series of arithmetical operations —addition, subtraction, multiplication, division, involution, and evolution—if it be desired to have the final result limited to a certain number of significant figures, it is unnecessary to use more significant figures in any of the numbers operated on than the desired number in the result plus 1.* For example, if only four significant figures are desired in the final result, all the numbers used in the various operations may be limited to 4 + 1 = 5 significant figures, the fifth figure being increased by 1 in all cases if the sixth figure is 5 or a greater digit.

From the foregoing, it follows that any method that will give five significant figures of the root correctly will be sufficiently exact for all practical purposes. Such a method will now be explained for extracting square root.

263. Suppose it is desired to find the square root of 20; that is, $\sqrt{20} = ?$ The problem is to divide 20 into two equal factors, or into two factors, the first five significant figures of which shall be equal. Since 20 is not a perfect square, inspection shows that one of the equal factors is 4 plus an interminable decimal, since 20 lies between $4^2 = 16$ and $5^2 = 25$. Dividing 20 by 4, the result is 5; i. e., $4 \times 5 = 20$. Now, by taking the average of these unequal factors, a new factor will be obtained, which will be nearer the correct value of the root than either of the two unequal factors, viz., $\frac{4+5}{2} = 4.5$, the square of which is $4.5^2 = 20.25$.

Assuming 4.5 for a new factor and dividing 20 by it, the result is $20 \div 4.5 = 4.444+$; that is, $4.444 \times 4.5 = 20$, nearly, the product not being exactly equal to 20 because 4.444 was used as one factor, instead of $4\frac{4}{9}$, the exact value. Again, taking the average of the two factors, $\frac{4.444 + 4.5}{2} = 4.472$, which is the root correct to at least three figures.

Assuming 4.47 to be one of the factors and dividing 20 by it to obtain the other, the result is $20 \div 4.47 = 4.474272+$; that is, $4.47 \times 4.474273 = 20$, very nearly. The average of these two factors is $\frac{4.47 + 4.474272}{2} = 4.472136+ = 4.4721$ to five significant figures. The exact root to 13 figures is 4.472135954999+.

That 4.4721 is the square root of 20 correct to five figures may easily be proved by squaring it; thus, $4.4721^2 = 19.99967841$, or 20.000 to five figures. Since the square agrees with the given number to five figures, the root is correct to five figures.

264. A close examination of the foregoing results reveals some remarkable facts. (1) The value of the first average 4.5 is correct to two figures of the root. (2) The value of the second average 4.472+ is correct to four figures of the root. (3) The value of the third average 4.472136 is correct to seven figures of the root. (4) All these averages are somewhat greater than the correct value of the root. (5) Of the two factors used in finding the average, one is a little greater and the other a little less than the correct value of the root. (6) Each step of the process gives a result approaching more and more nearly to the correct value of the root.

265. Calling the first average value the **first approximation,** the second average value the **second approximation,** and the third average value the **third approximation,** the following general method of procedure may be adopted: *Calculate the first approximation to two significant figures; the second approximation to three significant figures; and the third approximation to five significant figures. It is not safe to calculate the second approximation to more than three significant figures, because the fourth figure cannot, as a rule, be depended on. If the second significant figure of the first approximation be determined correctly, the third approximation will always be correct to at least five significant figures.*

266. The method will now be applied to numbers in general, and the best manner of explaining it is by means of examples.

EXAMPLE.— $\sqrt{714{,}627} = ?$

SOLUTION.—The first step is to point off the number into periods of two figures each, obtaining 71'46'27. To find the first approximation, only the first two significant figures are necessary; in this case, the first period, 71. The first figure of the root is evidently 8, since $8^2 = 64$ and $9^2 = 81$. The two factors then are 8 and $71 \div 8 = 8.87+$. The first approximation is $\frac{8 + 8.87}{2} = 8.43+ = 8.4$ to two figures.

To find the second approximation, use the first two periods, or 7146, and multiply the result obtained for the first approximation by 10. One factor is then 84 and the other $7146 \div 84 = 85.07+$. The second approximation is therefore $\frac{84 + 85.07}{2} = 84.53+$, or 84.5 to three figures.

To find the third approximation, use the first three periods, or 714627, and multiply the result obtained for the second approximation by 10. One factor is then 845 and the other is $714627 \div 845 = 845.712+$. The third approximation is therefore $\frac{845 + 845.712}{2} = 845.356$, or 845.86 to five significant figures. Ans.

REMARK.—It will be noticed in the last example, and also in those that follow, that when finding the unknown factor to be used in determining the value of the first, second, or third approximation, the division is carried one place farther than the number of figures desired in the approximation and that no attention is paid to the succeeding figures. Thus, in the last example, 71 ÷ 8 = 8.875, or 8.88, correct to three figures, while the number used was 8.87. The reason for this is that the value obtained for the approximation would be the same in either case, and it saves time to calculate as here shown. For instance, using 8.88 for the second factor, the first approximation is $\frac{8 + 8.88}{2} = 8.44$, or 8.4 to two figures.

267. The decimal point is located by employing the following principle: *There must be as many figures in the integral* part of the root as there are periods in the integral part of the given number whose root is to be found. If the given number is wholly decimal and there are two or more ciphers between the decimal point and the first significant figure, there will be as many ciphers between the decimal point and the first significant figure of the root as there are entirely cipher periods between the decimal point and the first significant figure of the given number.* Had the number in the last example been 71.4627, the root would have been 8.4536; had it been .714627, the root would have been .84536; had it been .0000714627, the root would have been .0084536. In the latter case, the number would have been pointed off thus, .00'00'71'46'27.

268. In all cases, numbers having the same significant parts and the same number of significant figures in the first (or left-hand) period of the significant part of the number, have the same significant figures in the root, the roots differing only in the position of the decimal point.

EXAMPLE.— $\sqrt{714.627} = ?$

SOLUTION.—Pointing off into periods, we have 7'14.62'70, adding a cipher to complete the last period. In all cases when pointing off the decimal part of numbers, begin at the decimal point and point off to the right, and add ciphers to the last period when it does not contain enough figures to make up a period. Since the first period contains but one figure and it is necessary to have two figures at least in order that the first approximation may be correct to two figures, regard the decimal point as situated between 7 and 1 instead of between 4 and 6, thus obtaining 7.1 for the first two figures of the given number. The first two figures of the square root of 7.1 will be the same as the first two figures of the square root of 714.

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* The *integral* part of a number is the part to the left of the decimal point. Thus, the integral part of 1,726.943 is 1,726.

It is evident that the first figure of the root is 2, since $2^2 = 4$ and $3^2 = 9$. Using 2 as one factor, the other is $7.1 \div 2 = 3.55$, and the first approximation is $\frac{2 + 3.55}{2} = 2.77+$, or 2.8 to two figures. Had 3 been used as one factor, the other would have been $7.1 \div 3 = 2.36+$, and the first approximation would have been $\frac{3 + 2.36}{2} = 2.68$, or 2.7 to two figures. In the first case, the difference between the two factors is $3.55 - 2 = 1.55$; in the second case, the difference is $3 - 2.36 = .64$. As the factors are more nearly equal in the second case than in the first, it is evident that 2.7 is more nearly equal to the correct value of the root than 2.8 is; hence, 2.7 will be used for the first approximation.

For the second approximation, use the first two periods and 27 for one factor, the other factor being $715 \div 27 = 26.48+$; hence, the second approximation $= \frac{27 + 26.48}{2} = 26.74$, or 26.7 to three figures. We used 715 for the first three figures of the given number, instead of 714, because the fourth figure was 6 and the number correct to three figures was 715. In finding the third approximation, the first three periods may be used or all the figures; the result will be the same in either case. Since it is better to use six figures than five, move the decimal point two places to the right, obtaining 71462.7; one factor is 267 and the other $71462.7 \div 267 = 267.650+$. The third approximation is $\frac{267 + 267.650}{2} = 267.325$, or 267.33 to five figures. Since there are two periods in the integral part of the given number, there are two figures in the integral part of the root, and $\sqrt{714.627} = 26.733$. Ans.

269. When determining the first approximation, that number should always be used for the first factor which will make the less difference between it and the second factor, as was done in the last example. Thus, for 2.5, the factors would be 2 and 1.25, the difference between them being .75. If 1 were selected for the first factor, the other would be $2.5 \div 1 = 2.5$, and the difference between them 1.5. In one case, the first approximation would be $\frac{2 + 1.25}{2} = 1.6+$, and in the other case, $\frac{1 + 2.5}{2} = 1.8-$. Since $1.6^2 = 2.56$ and

$1.8^2 = 3.24$, it is evident 1.6 is very much nearer the correct value of the root than 1.8.

270. If the given number is a perfect square and contains not more than ten significant figures, the exact root will be obtained in all cases. That the number is a perfect square may be suspected by the fact that there are one or more 9's or 0's following the fifth figure of the number expressing the third approximation, and that when the third approximation is expressed correct to five figures, the square of its last digit (or the second figure of this square when the square contains more than one figure) will be the same as the last digit of the given number. This will be illustrated by two examples.

EXAMPLE.— $\sqrt{3,749,602,756} = ?$

SOLUTION.—Pointing off, we obtain 37'49'60'27'56. The first two factors are evidently 6 and $37 \div 6 = 6.16+$, and the first approximation $\frac{6 + 6.16}{2} = 6.08$, or 6.1 to two figures.

$$3749 \div 61 = 61.45+;\ \frac{61 + 61.45}{2} = 61.23-, \text{ or } 61.2 \text{ to three figures.}$$

$$374960 \div 612 = 612.67973+;\ \frac{612 + 612.67973}{2} = 612.33986+,$$

or 612.34 to five figures. But $4^2 = 16$, and as the last figure of the given number is also 6, and as the sixth and seventh figures are 9 and 8, respectively, we suspect that the given number is a perfect power. It may not be, however, for the reason that the figures 5, 7, and 2, preceding 6, may be different from the ones given without changing the value of the root to five figures. Hence, the only way to ascertain the fact beyond possibility of doubt is to square the root; doing so, it is found that $61,234^2 = 3,749,602,756$, which is therefore a perfect square.

Had all the figures of the given number been used in finding the third approximation, the result would have been as follows: $3,749,602,756 \div 612 = 612.6801+$, and $\frac{612 + 612.6801}{2} = 612.34005+$, or 612.34 to five figures, as before, or 61,234 after locating the decimal point. Ans.

271. If the given number contains not more than three periods of significant figures—that is, if it contains not more than five or six significant figures—and is a perfect power, the fact will be revealed when finding the second factor in the third approximation, for the two factors will then be exactly equal.

EXAMPLE.— $\sqrt{.00095481} = ?$

SOLUTION.— .00095481 = .00'09'54'81 when pointed off into periods of two figures each. The first two significant figures are 9.5. The first factor is evidently 3 and the second factor 9.5 ÷ 3 = 3.16+. The first approximation is $\frac{3 + 3.16}{2} = 3.08$, or 3.1 to two figures.

955 ÷ 31 = 30.80+; the second approximation is $\frac{31 + 30.80}{2} = 30.90$ = 30.9 to three figures.

95481 ÷ 309 = 309; hence, .00095481 is a perfect power and the significant figures of the root are 309. There being one full cipher period following the decimal point, the root is .0309. Ans.

272. One more example will be given to show the student how to arrange his work when solving examples in square root by this method.

EXAMPLE.— $\sqrt{3,265.47} = ?$

SOLUTION.— 3,265.47 = 32'65.47.

33 ÷ 5 = 6.6; 33 ÷ 6 = 5.5; 6.6 − 5 = 1.6; 6 − 5.5 = .5; hence, use 6 for first factor.

$$\frac{6 + 5.5}{2} = 5.75, \text{ or } 5.8.$$

$$3,265 \div 58 = 56.29; \frac{58 + 56.29}{2} = 57.14, \text{ or } 57.1.$$

326,547 ÷ 571 = 571.886; $\frac{571 + 571.886}{2} = 571.443$, or 571.44 to five figures. Therefore, $\sqrt{3,265.47} = 57.144$. Ans.

EXAMPLES FOR PRACTICE.

Find the square root of:

			Ans.
(*a*)	186,624.	(*a*)	432.
(*b*)	2,050,624.	(*b*)	1,432.
(*c*)	29,855,296.	(*c*)	5,464.
(*d*)	.0116964.	(*d*)	.10815−.
(*e*)	198.1369.	(*e*)	14.076+.
(*f*)	994,009.	(*f*)	997.
(*g*)	2.375.	(*g*)	1.5411+.
(*h*)	1.625.	(*h*)	1.2748−.
(*i*)	.3025.	(*i*)	.55.
(*j*)	.571428.	(*j*)	.75593−.
(*k*)	.78125.	(*k*)	.88388+.

CUBE ROOT.

273. Cube root may be extracted in a manner similar to that just described for square root, the only essential differences being that the given number must be pointed off into periods of *three* figures each; the first period, if integral, may contain one, two, or three figures; and the number must be divided into three equal factors.

274. As might be expected, cube root is a longer operation than square root, but the method is similar and is no more difficult to remember or apply. As in the case of square root, it is unnecessary to use more than six significant figures in order to obtain five significant figures of the root. The method is best illustrated by an example. The student is advised to make a little table, containing the cubes of numbers from 1 to 9, similar to that here given.

$1^3 = 1$
$2^3 = 8$
$3^3 = 27$
$4^3 = 64$
$5^3 = 125$
$6^3 = 216$
$7^3 = 343$
$8^3 = 512$
$9^3 = 729$

EXAMPLE.— $\sqrt[3]{389{,}247} = ?$

SOLUTION.— $389{,}247 = 389'247$ when pointed off into periods of three figures each. As in the case of square root, consider the first period only when finding the first approximation. In other words, divide 389 into three factors as nearly equal as possible. It is readily seen that 389 lies between $7^3 = 343$ and $8^3 = 512$; hence, the first figure of the root is 7. Now, assume that two of the equal factors are each equal to 7 and divide 389 by their product to obtain the third factor; that is, divide 389 by $7^2 = 49$. The result is $389 \div 49 = 7.93+$. Hence, $7 \times 7 \times 7.93+ = 389$, nearly. The average of these factors is $\frac{7 + 7 + 7.93}{3} = \frac{2 \times 7 + 7.93}{3} = 7.31$, or 7.3 to two figures, the first approximation.

Assuming 73 to be the value of two of the three equal factors, divide the first two periods of the given number by their product 73×73; that is, by 73^2, or 5,329. The result is $389{,}247 \div 5{,}329 = 73.04+$; that is, $7.3 \times 7.3 \times 73.04 = 389{,}247$, nearly. The average of the three factors is $\frac{2 \times 73 + 73.04}{3} = 73.01$, or 73.0 to three figures, the second approximation.

Assuming 730 to be two of the three equal factors, divide 389,247 by 730^2, or by 73^2, since the cipher at the right is not a significant figure and will not affect the result, obtaining for the third factor 73.0431.

The average of these three factors is $\frac{2 \times 73 + 73.0431}{3} = 73.0143$, or 73.014 to five figures. Ans.

NOTE.—The decimal point is located by applying the principle of Art. **267**; viz., there must be as many figures in the integral part of the root as there are periods in the integral part of the given number.

275. An inspection of the foregoing example shows that about the only respect in which the work of extracting cube root exceeds the work of extracting square root consists in squaring one number of two figures and one number of three figures. The work of division in finding the third factor is a little harder on account of the divisors being a little larger than when finding the second factor in square root.

276. If the given number contains an integral part, it is better to locate the decimal point as soon as possible, in order to prevent confusion, instead of waiting until the third approximation has been found.

EXAMPLE.— $\sqrt[3]{3.274} = ?$

SOLUTION.—The first period contains but one figure; therefore, we operate on three figures in order to have the first approximation correct to two figures (see **c**, Art. **261**). If 1 be chosen for one of the two equal factors, the third factor will be $3.27 \div 1^2 = 3.27$, and the difference between one of the equal factors and the third factor will be $3.27 - 1 = 2.27$. If 2 be chosen for one of the equal factors, the third one will be $3.27 \div 2^2 = .817+$, and the difference between this and one of the equal factors is $2 - .817 = 1.183$. Since 1.18 is less than 2.27, use 2 for one of the two equal factors. The first approximation is $\frac{2 \times 2 + .817}{3} = 1.60+$, or 1.6 to two figures. Since there is but one period in the integral part of the given number, the root is equal to 1 plus an interminable decimal, as the given number is not a perfect cube. Therefore, retain the decimal point in its present position through all the subsequent operations.

Assuming 1.6 to be one of the two equal factors, the third factor is $3.274 \div 1.6^2 = 3.274 \div 2.56 = 1.278+$, and the second approximation is $\frac{2 \times 1.6 + 1.278}{3} = 1.492+$, or 1.49 to three figures.

Assuming 1.49 to be one of the two equal factors, the other factor is $3.274 \div 1.49^2 = 3.274 \div 2.2201 = 1.474708+$; hence, the third approximation is $\frac{2 \times 1.49 + 1.474708}{3} = 1.484902+$, or 1.4849 to five figures. Ans.

The exact root to seven figures is 1.484886—.

277. The only case in cube root that will give any trouble in determining the fifth significant figure correctly is when the difference between the numbers representing the first and second approximations, expressed to two figures, is greater than one unit in the second figure. In the last example, the first approximation was 1.6 and the second 1.49, or 1.5 to two figures; the difference is .1, or one unit in the second figure. For numbers the significant part of whose first period is 2, the difference between the first and second approximations may differ by more than one unit in the second figure; in such cases, recalculate the second approximation, using for one of the equal factors the value of the second approximation to two figures as first determined. An example will illustrate this.

EXAMPLE.— $\sqrt[3]{.0027} = ?$

SOLUTION.— .0027 = .002'700 when pointed off into periods. But, the significant figures in the cube root of 2.7 will be the same as in the cube root of .002'700; therefore, find the cube root of 2.7 and locate the decimal point after the operation is finished.

If 1 be chosen as one of the equal factors, the third factor will be $2.7 \div 1^2 = 2.7$, and the first approximation is $\frac{2 \times 1 + 2.7}{3} = 1.56+$, or 1.6 to two figures. If 2 be chosen for one of the equal factors, the third factor is $2.7 \div 2^2 = .675$, and the first approximation is $\frac{2 \times 2 + .675}{3} = 1.55+$, or 1.6 to two figures.

Using 1.6 for one of the equal factors, the third factor is $2.7 \div 1.6^2 = 2.7 \div 2.56 = 1.054+$, and the second approximation is $\frac{2 \times 1.6 + 1.054}{3} = 1.418$, or 1.42 to three figures, or 1.4 to two figures. The difference between the first and second approximations is $1.6 - 1.4 = .2$, or two units in the second figure. Therefore, recalculate the second approximation, using 1.4 for one of the equal factors. The third factor is then equal to $2.7 \div 1.4^2 = 2.7 \div 1.96 = 1.377$, and the second approximation is $\frac{2 \times 1.4 + 1.377}{3} = 1.392+$.

Using 1.39 for one of the equal factors, the third factor is $2.7 \div 1.39^2 = 2.7 \div 1.9321 = 1.39744+$, and the third approximation is $\frac{2 \times 1.39 + 1.39744}{3} = 1.39248$, or 1.3925 to five figures. The root correct to nine figures is 1.39247665. Since the given number is wholly decimal and has no period composed entirely of ciphers, $\sqrt[3]{.0027} = .13925$. Ans.

Had 1.42 been used for one of the equal factors, the third approximation would have been 1.3930.

278. The remarks made in Art. **270** regarding the square root of perfect squares apply, with slight modifications, to the cube root of perfect cubes. If the given number is a perfect cube and contains not more than five periods, i. e., not more than $5 \times 3 = 15$ significant figures, the exact root can always be found. That the given number is a perfect cube will be suspected from the fact that the root ends in a string of 9's or 0's; that in one of the approximations the three factors become exactly equal; and that the last digit in the cube of the last figure of the root is the same as the last digit of the given number. An example will illustrate this.

EXAMPLE.— $\sqrt[3]{106{,}294{,}343.553} = ?$

SOLUTION.—The number when pointed off becomes 106'294'343.553; hence, there are three figures in the integral part of the root. The first period 106 lies between $4^3 = 64$ and $5^3 = 125$. Trying 4 for one of the equal factors, the third factor is $106 \div 4^2 = 6.62+$. Trying 5, the third factor is $106 \div 5^2 = 4.24$; hence, use 5, and obtain for the first approximation $\frac{2 \times 5 + 4.24}{3} = 4.74+$. Using two periods and 47 for one of the equal factors, the third factor is $106{,}294 \div 47^2 = 106{,}294 \div 2{,}209 = 48.11+$, and the second approximation is $\frac{2 \times 47 + 48.11}{3} = 47.37$.

To find the third approximation, two, or three, or all four periods may be used, since the first two periods contain six significant figures, and hence will give the root correct to five figures (see **c**, Art. **261**). Using the first three periods, to avoid the decimal point, and 474 for one of the equal factors, the third factor is $106{,}294{,}343 \div 474^2 = 106{,}294{,}343 \div 224{,}676 = 473.1005+$, and the third approximation is $\frac{2 \times 474 + 473.1005}{3} = 473.7001+$, or 473.70 to five figures. It will be noticed that the results obtained for the second and third approximations are alike and the last digit in $7^3 = 343$ is the same as the last significant figure of the given number; hence, it is at once suspected that the given number is a perfect power, and this is proved by cubing the root. Therefore, $\sqrt[3]{106{,}294{,}343.553} = 473.7$. Ans.

279. Square and cube root are two of the most important operations described in Arithmetic, and the student is earnestly advised to thoroughly familiarize himself with the process. Few practical problems involving mensuration arise that do not require the extraction of the square or cube root. For instance, to find the diameter of a circle that will contain a given area requires the extraction of square root; to find the diameter of a sphere that will contain a given volume requires the extraction of cube root.

EXAMPLES FOR PRACTICE.

Find the cube root of:

(*a*)	78,347.809639.	Ans.	(*a*)	42.79.
(*b*)	2.		(*b*)	1.2599+.
(*c*)	4,180,769,192.462.		(*c*)	1,611.0–.
(*d*)	.696.		(*d*)	.88621–.
(*e*)	.375.		(*e*)	.72112+.
(*f*)	513,229.783302144.		(*f*)	80.064.

TABLE METHOD OF EXTRACTING SQUARE AND CUBE ROOT.

280. By means of the table of Squares, Cubes, Fourth, and Fifth Powers, which contains the squares and cubes of numbers from 1 to 10, varying by tenths, and the first five figures of the fourth and fifth powers of the same numbers, the first three, and frequently the first four, significant figures of the square root or cube root of any number can be readily determined. The remaining figures can then be easily determined in the same manner as the third approximation in the preceding pages.

The student is advised to use the table in all cases, as it will greatly shorten his work.

281. By the aid of this table the first two significant figures of the root can be obtained directly and one more by a slight calculation. For example, suppose it is desired to find the first three significant figures of $\sqrt{5,269.73}$. Pointing off into periods and moving the decimal point so that it falls between the first and second periods, the number becomes 52.69'73; in other words, the significant figures of $\sqrt{5,269.73}$ are the same as for $\sqrt{52.6973}$. Since four figures only are given in the table, reduce the given number to four figures. The problem then becomes: find the first three figures of $\sqrt{52.70}$. Referring to the table, 52.70 lies between

SQUARES, CUBES, FOURTH, AND FIFTH POWERS.

No.	Square.	Cube.	4th Power.	5th Power.	No.	Square.	Cube.	4th Power.	5th Power.
1.0	1.00	1.000	1.0000	1.0000	5.5	30.25	166.375	915.06	5 032.8
1.1	1.21	1.331	1.4641	1.6105	5.6	31.36	175.616	983.45	5.507.3
1.2	1.44	1.728	2.0736	2.4883	5.7	32.49	185.193	1,055.6	6,016.9
1.3	1.69	2.197	2.8561	3.7129	5.8	33.64	195.112	1,131.6	6,563.6
1.4	1.96	2.744	3.8416	5.3782	5.9	34.81	205.379	1,211.7	7,149.2
1.5	2.25	3.375	5.0625	7.5938	6.0	36.00	216.000	1,296.0	7,776.0
1.6	2.56	4.096	6.5536	10.486	6.1	37.21	226.981	1,384.6	8,446.0
1.7	2.89	4.913	8.3521	14.199	6 2	38.44	238.328	1,477.6	9,161.3
1.8	3.24	5.832	10.498	18.896	6.3	39.69	250.047	1,575.3	9,924.4
1.9	3.61	6.859	13.032	24.761	6.4	40.96	262.144	1,677.7	10,737
2.0	4.00	8.000	16.000	32.000	6.5	42 25	274.625	1,785.1	11,603
2.1	4.41	9.261	19.448	40.841	6.6	43.56	287.496	1,897.5	12,523
2.2	4.84	10.643	23.426	51.536	6.7	44.89	300.763	2,015.1	13,501
2.3	5.29	12.167	27.984	64.363	6.8	46.24	314.432	2,138.1	14,539
2.4	5.76	13.824	33.178	79.626	6.9	47.61	328.509	2,266.7	15,640
2.5	6.25	15.625	39.063	97.656	7.0	49 00	343.000	2,401.0	16,807
2.6	6.76	17.576	45.698	118.81	7.1	50.41	357.911	2,541.2	18,042
2.7	7.29	19.683	53.144	143.49	7.2	51.84	373.248	2,687.4	19,349
2.8	7.84	21.952	61.466	172.10	7.3	53.29	389.017	2,839.8	20,731
2.9	8.41	24.389	70.728	205.11	7.4	54.76	405.224	2,998.7	22,190
3.0	9.00	27.000	81.000	243.00	7.5	56.25	421.875	3,164.1	23.730
3.1	9 61	29.791	92.352	286.29	7.6	57.76	438.976	3,336.2	25,355
3.2	10.24	32.768	104.86	335.54	7.7	59.29	456.533	3,515.3	27,068
3.3	10.89	35.937	118.59	391.35	7.8	60.84	474.552	3,701.5	28,872
3.4	11.56	39.304	133.63	454.35	7.9	62.41	493.039	3,895.0	30,771
3.5	12.25	42.875	150.06	525.22	8.0	64.00	512.000	4,096.0	32,768
3.6	12.96	46.656	167.96	604.66	8.1	65.61	531.441	4,304.7	34,868
3.7	13.69	50.653	187.42	693.44	8.2	67.24	551.368	4,521.2	37,074
3.8	14.44	54.872	208.51	792.35	8.3	68.89	571.787	4,745.8	39,390
3.9	15.21	59.319	231.34	902.24	8.4	70.56	592 704	4,978.7	41,821
4.0	16.00	64.000	256.00	1,024.0	8.5	72.25	614.125	5,220.1	44,371
4.1	16.81	68.921	282.58	1,158.6	8.6	73.96	636.056	5,470.1	47,043
4.2	17.64	74.088	311.17	1,306.9	8.7	75.69	658.503	5,729.0	49,842
4.3	18.49	79.507	341.88	1,470.1	8.8	77.44	681.472	5,997.0	52,778
4.4	19.36	85.184	374.81	1,649.2	8.9	79.21	704.969	6,274.2	55,841
4.5	20.25	91.125	410.06	1,845.3	9.0	81.00	729.000	6,561.0	59,049
4.6	21.16	97.336	447.75	2,059.6	9.1	82.81	753.571	6,857.5	62,403
4.7	22.09	103.823	487.97	2,293.5	9.2	84.64	778.688	7,163.9	65,908
4.8	23.04	110.592	530.84	2 548.0	9.3	86.49	804.357	7,480.5	69,569
4.9	24.01	117.649	576.48	2,824.8	9 4	88.36	830.584	7,807.5	73,390
5.0	25.00	125.000	625.00	3,125.0	9 5	90.25	857.375	8,145.1	77,378
5.1	26.01	132.651	676.52	3,450.3	9.6	92.16	884.736	8,493.5	81,537
5.2	27.04	140.608	731.16	3 802.0	9.7	94.09	912.673	8,852.9	85,873
5.3	28.09	148.877	789.05	4,182.0	9.8	96.04	941.192	9,223.7	90,392
5.4	29.16	157.464	850.31	4,591.7	9.9	98.01	970.299	9,606.0	95,099

$51.84 = 7.2^2$ and $53.29 = 7.3^2$; hence, the first two figures of the root are 7.2. Find the difference between the two numbers in the table between which the given number falls and call it the **first difference**; thus, $53.29 - 51.84 = 1.45 =$ the first difference. Find the difference between the lower number in the table and the given number and call it the **second difference**; thus, $52.70 - 51.84 = .86 =$ the second difference. Divide the second difference by the first difference, and the first figure of the quotient, if the quotient is .05 or greater, will be the third figure of the root, when reduced to one figure. If the quotient is less than .05, the third figure of the root is a cipher. Thus, $.86 \div 1.45 = .59+$, or .6 when reduced to one figure. Therefore, the first three figures of $\sqrt{52.70}$ are 7.26. Since the integral part of the given number contains two periods, there are two figures in the integral part of the root; therefore, $\sqrt{5{,}269.73} = 72.6$ to three figures. Ans.

282. The cube root is found to three significant figures in exactly the same way, as shown in the following example:

EXAMPLE.—Find the first three figures of $\sqrt[3]{.0625}$.

SOLUTION.—Pointing off and placing the decimal point between the first and second significant periods, the result is 62.500. Referring to the table, the first two figures of the root are 3.9; the first difference is $64.000 - 59.319 = 4.681$; the second difference is $62.500 - 59.319 = 3.181$; $3.181 \div 4.681 = .67+$, or .7 to one figure. Therefore, $\sqrt[3]{62.5} = 3.97$, and $\sqrt[3]{.0625} = .397$ to three significant figures. Ans.

283. Having found the first three significant figures by means of the table, find the fourth and fifth figures in the usual manner by using the first three figures in finding the third approximation.

For example, find the cube root of 126.57 to five figures. Referring to the table, the first two figures are 5.0. The first difference is $132.651 - 125.000 = 7.651$; the second difference is $126.57 - 125.000 = 1.57$; $1.57 \div 7.651 = .20+$. Hence, the first three figures are 5.02. Using 5.02 for one of the equal factors, the third factor is $126.57 \div 5.02^2$

$= 5.02253+$, and the third approximation is $\frac{2\times 5.02+5.02253}{3}$ $= 5.02084+$, or $\sqrt[3]{126.57} = 5.0208$ to five figures. Ans.

284. If more than five significant figures of the square or cube root are desired, use the five figures of the third approximation for one of the equal factors and calculate the unknown factor to as many figures as are desired plus one; the next approximation will be correct to at least nine figures, if the unknown factor has been calculated to ten figures.

ROOTS OF FRACTIONS.

285. If the given number is in the form of a fraction, and it is required to find some root of it, the simplest and most exact method is to reduce the fraction to a decimal and extract the required root of the decimal. If, however, the numerator and denominator of the fraction are perfect powers, extract the required root of each separately, and write the root of the numerator for a new numerator and the root of the denominator for a new denominator.

Example 1.—What is the square root of $\frac{9}{64}$?

Solution.— $\sqrt{\frac{9}{64}} = \frac{\sqrt{9}}{\sqrt{64}} = \frac{3}{8}$. Ans.

Example 2.—What is the square root of $\frac{5}{8}$?

Solution.— $\sqrt{\frac{5}{8}} = \sqrt{.625} = .79057-$, since $\frac{5}{8} = .625$. Ans.

Example 3.—What is the cube root of $\frac{27}{64}$?

Solution.— $\sqrt[3]{\frac{27}{64}} = \frac{\sqrt[3]{27}}{\sqrt[3]{64}} = \frac{3}{4}$. Ans.

Example 4.—What is the cube root of $\frac{1}{4}$?

Solution.—Since $\frac{1}{4} = .25$, $\sqrt[3]{\frac{1}{4}} = \sqrt[3]{.25} = .62996+$. Ans.

286. Rule.—*Extract the required root of the numerator and denominator separately; or, reduce the fraction to a decimal and extract the root of the decimal.*

EXAMPLES FOR PRACTICE.

(a) $\sqrt{\frac{9}{16}} = ?$

(b) $\sqrt[3]{\frac{27}{1728}} = ?$

(c) $\sqrt[3]{\frac{27}{375}} = ?$

(d) $\sqrt[3]{\frac{560}{128}} = ?$

Ans. (a) $\frac{3}{4}$. (b) $\frac{1}{4}$. (c) .41602–. (d) 1.6355+.

FOURTH ROOT.

287. The fourth root may be found by a method similar to that just described for extracting square and cube roots, dividing the given number into periods of four figures each and resolving the given number into four equal factors. It is generally easier and shorter, however, to extract the square root and then extract the square root of the result. For example, to extract the fourth root of 5,735,796,283.8016, which is a perfect fourth power (and consequently a perfect square, also), the square root would be extracted in the usual manner, obtaining 75,735.04. The square root of this result would then be extracted, obtaining 275.2. In other words $\sqrt[4]{5,735,796,283.8016} = \sqrt{\sqrt{5,735,796,283.8016}} = \sqrt{75,735.04} = 275.2$.

The fourth root is very seldom required and can always be found as just described.

FIFTH ROOT.

288. The fifth root is required oftener than the fourth root, but nevertheless it is seldom necessary to extract it. The method is the same in principle as that explained for cube root. The given number is divided into periods of five figures each and resolved into five equal factors; the first period may contain one, two, three, four, or five figures. As in the case of cube root, it is advisable to construct a little table giving the fifth powers of the nine digits, similar to that here given. An example will illustrate the process.

$1^5 = 1$
$2^5 = 32$
$3^5 = 243$
$4^5 = 1,024$
$5^5 = 3,125$
$6^5 = 7,776$
$7^5 = 16,807$
$8^5 = 32,768$
$9^5 = 59,049$

Example.— $\sqrt[5]{5,186.42} = ?$

Solution.—The first period 5186 lies between $5^5 = 3125$ and $6^5 = 7776$; hence, the root is 5 plus an interminable decimal. Trying 5 as one of the four equal factors and dividing the first period by their product to find the fifth factor, we have $\frac{5186}{5 \times 5 \times 5 \times 5} = 5186 \div 5^4 = 5186 \div 625 = 8.29+$. Trying 6 as one of the four equal factors, the fifth factor is $5186 \div 6^4 = 5186 \div 1296 = 4.00+$. Since the difference between 6 and 4 is less than the difference between 5 and 8.29, use 6 as one of the equal factors. Then, $5186 = 6 \times 6 \times 6 \times 6 \times 4$. The average of these factors is $\frac{6+6+6+6+4}{5} = \frac{4 \times 6 + 4}{5} = \frac{28}{5} = 5.6$, the first approximation.

Using 5.6 as one of the equal factors, the fifth factor is $5186.42 \div 5.6^4 = 5186 \div 983.4496 = 5186 \div 983.4000$ (using but four significant figures, since only three figures of the second approximation are required—see **b,** Art. **261**) $= 5.273$, and the second approximation is $\frac{4 \times 5.6 + 5.273}{5} = 5.534+$, or 5.53 to three figures.

Using 5.53 as one of the equal factors, the fifth factor is $5186.42 \div 5.53^4 = 5186.42 \div 935.191 = 5.54584+$, and the third approximation is 5.53316+, or 5.5332 to five figures. Ans.

The exact root to seven figures is 5.533164.

289. In order that the fifth significant figure of the fifth root of a number may be correct, it is absolutely essential that the third significant figure of the second approximation be correct. In the last example, it will be noticed that the difference between the first and second approximations is $5.60 - 5.53 = .07$, which is less than one unit in the second figure, but very near to it. Had this difference been as much as 1, or had it exceeded one unit in the second figure, it would have been advisable to recalculate the second approximation.

290. The labor involved in extracting the fifth root is very much greater than that necessary to extract the cube root, chiefly on account of raising numbers to the fourth power. This labor may be shortened considerably in the following manner:

Consider any number, as 4. Now, $4^4 = 4 \times 4 \times 4 \times 4 = (4 \times 4) \times (4 \times 4) = 16 \times 16 = 256$. In other words, to raise a number to the fourth power, square the number and then square the square. Now, the square of any number contains twice as many significant figures as the number or twice as many less 1; the cube of any number contains three times as many significant figures as the number or three times as many less one or two; the fourth power will contain four times as many or four times as many less one, two, or three; and so on. Hence, the fourth power of a number containing two figures will contain five, six, seven, or eight figures; and of one containing three figures, nine, ten, eleven, or twelve figures. In determining the fifth factor for the second approximation, only four figures of the fourth power are required, and in determining the fifth factor for the third approximation, only six figures of the fourth power are required. Therefore, any method that will enable us to dispense with unnecessary figures will lessen the work. The following method, which will also apply to any case of multiplication when only a certain definite number of figures are desired in the product, is the best we know of; it is best illustrated by an example.

Example.—Multiply 467,295 by 634,187 and obtain six figures of the product correct.

Solution.—

(*a*)		(*b*)
467295		4'6'7'2'9'5
634137		634137
2803770		2803770
140188	5	140188
18691	80	18691
467	295	467
140	1885	140
32	71065	32
2963290	49415	2963288, or **296329** to six figures.

Explanation.—Instead of beginning with the last, or right-hand, digit, as in ordinary multiplication, begin with the first, or left-hand, digit of the multiplier and multiply in the ordinary manner, the product being 2,803,770. Multiply by the second digit and write the first figure obtained in the second partial product one place to the right of the last figure of the first partial product. So proceed until all the partial products have been found, adding them to find the entire product. The result is shown at (*a*). Now, in order to have six figures of the product correct, seven figures should be obtained (see **a**, Art. **261**). It is therefore evident that all figures to the right of the vertical line in (*a*) are unnecessary. Hence, proceed as shown in (*b*). The first partial product contains seven figures—all that are required; therefore, cut off the figure 5 in the multiplicand when finding the second partial product, but multiply it by 3 in order to determine how much to carry. Thus, say mentally "three times five is fifteen," and carry 1. Then say "three times nine is twenty-seven and one is twenty-eight," etc. When multiplying by the next digit 4, cut off the second figure from right of the multiplicand, but carry the 3 that is obtained by multiplying 9 by 4, and say "four times two is eight and three is eleven." Proceeding in this manner, no figure of any of the partial products will extend beyond the place occupied by the seventh figure of the entire product.

291. The operation of division may be shortened in a similar manner to that just described for multiplication. Perform the division in the usual manner until the number of significant figures in the quotient equals the number obtained by subtracting the number of significant figures in the divisor from the number desired in the quotient plus three; then cut off one figure from the right of the divisor before finding the next figure of the quotient; cut off the second figure from the right of the divisor before finding the succeeding figure of the quotient; and so on until the quotient contains one more than the required number of figures. It is here assumed that the dividend and divisor do not contain more than one significant figure in excess of the number required in the quotient. (See **a** and **b**, Art. **261**.)

EXAMPLE.—Divide 71,346.247 by 27,846.392 and obtain five significant figures of the quotient correct.

Solution.—

$$\begin{array}{rl}
71846.20 & \big|\,\underline{2\,7'8'4'6'.4} \\
556928 & \;\;2.56213\text{, or }2.5621\text{ to five figures.}\quad\text{Ans.} \\ \cline{1-1}
1565340 & \\
1392320 & \\ \cline{1-1}
173020 & \\
167078 & \\ \cline{1-1}
5942 & \\
5569 & \\ \cline{1-1}
373 & \\
278 & \\ \cline{1-1}
95 & \\
83 & \\ \cline{1-1}
12 &
\end{array}$$

Explanation.—Since five significant figures are required in the quotient, the dividend and divisor are limited to six significant figures. The number of significant figures required in the quotient before beginning to cut off figures from the divisor is $5 + 3 - 6 = 8 - 6 = 2$; hence, before finding the third figure of the quotient, cut off the figure 4 from the right of the divisor, but multiply 4 by 6 in order to see how much to carry. Thus, say "six times four is twenty-four," and carry 2; then, say "six times six is thirty-six and two is thirty-eight," and write 8 and carry 3; and so on. Before finding the fourth figure of the quotient, cut off the next figure 6 of the divisor. The student will find it convenient to place the divisor at the right of the dividend with the quotient underneath, as shown. This arrangement saves space and brings each figure of the quotient directly under the divisor, making the multiplication easier.

292. To locate the decimal point in the quotient, the easiest way is to proceed as follows: Move the decimal point in the divisor to the right until it follows the right-hand figure; that is, make the divisor a whole number; move the decimal point in the dividend as many decimal places to the right as it was moved in the divisor, annexing ciphers if necessary. If the dividend will contain the divisor one or more times, there will be as many figures in the integral part of the quotient as there are figures left in the dividend after finding the first remainder plus one. If the dividend will not contain the divisor, annex ciphers to follow the decimal point until the dividend contains the divisor, and the first significant figure of the quotient will then be located as many decimal places to the right of the decimal point as there were ciphers annexed. For instance, $.046 \div 21.76 = 4.6 \div 2176. = 4.600 \div 2{,}176. = .002+$; $4.6 \div 21.76 = 460. \div 2{,}176. = 460.0 \div 2{,}176 = .2+$; $460 \div 21.76 = 46{,}000. \div 2{,}176 = 21.+$. In the last example, $71{,}346.2 \div 27{,}846.4 = 713{,}462. \div 278{,}464. = 2+$.

293. Having shown how the work of calculation may be greatly reduced, an example is given on the following page, showing all the figures used in extracting the fifth root, each operation being numbered in the order in which it is performed.

294. When the sixth significant figure of the third approximation is 5, it is not always advisable to increase the fifth figure by one. To ascertain whether or not the fifth figure should be increased, recalculate the third approximation, using for one of the equal factors the third approximation first found, correct to *four* figures; if the sixth significant figure is then 5+, increase the fifth figure by 1.

EXAMPLE 1.— $\sqrt[5]{3.056} = ?$

SOLUTION.—Using the first two significant figures and trying 1 for the equal factors, the fifth factor is $3.1 \div 1^4 = 3.1$, and the first approximation is $\frac{4 \times 1 + 3.1}{5} = 1.42$. Since $3.1 - 1 = 2.1$ is less than 2.5 (see Remark in example referred to in Art. **293**), it is not necessary to try 2 for one of the equal factors.

$$3.056 \div 1.4^4 = .795+. \quad \frac{4 \times 1.4 + .795}{5} = 1.279+.$$

Since the difference between 1.42, the first approximation, and 1.279 the second approximation, is greater than one unit in the second figure, try 1.3 for one of the equal factors and recalculate the second approximation.

$$3.056 \div 1.3^4 = 1.069+. \quad \frac{4 \times 1.3 + 1.069}{5} = 1.253+.$$

EXAMPLE.— $\sqrt[5]{8{,}269} = ?$

SOLUTION.—

(1)

$$\begin{array}{r} 6\\ 6\\ \hline 36\\ 36\\ \hline 108\\ 216\\ \hline 1296 \end{array}$$

(2)

$$\begin{array}{r|l} 8269 & 1296\\ \cline{2-2} 7776 & \quad 6.38+\\ \cline{1-1} 493 & \\ 388 & \\ \cline{1-1} 105 & \\ 103 & \\ \cline{1-1} 2 & \end{array}$$

(3)

$$\frac{4 \times 6 + 6.38}{5} = 6.07+$$

(6)

$$\frac{4 \times 6.1 + 5.974}{5} = \frac{30.374}{5} = 6.074+$$

(4)

$$\begin{array}{r} 6.1\\ 6.1\\ \hline 366\\ 61\\ \hline 37.21\\ 37.21\\ \hline 11163\\ 2604\\ 74\\ 3\\ \hline 1384.4 \end{array}$$

(5)

$$\begin{array}{r|l} 8269.0 & 1384\\ \cline{2-2} 6920 & \quad 5.974\\ \cline{1-1} 13490 & \\ 12456 & \\ \cline{1-1} 1034 & \\ 968 & \\ \cline{1-1} 66 & \\ 55 & \\ \cline{1-1} 11 & \end{array}$$

(7)

$$\begin{array}{r} 6.07\\ 6.07\\ \hline 3642\\ 4249\\ \hline 36.8449\\ 36.'8'4'5\\ 36.845\\ \hline 110535\\ 221070\\ 29476\\ 1473\\ 184\\ \hline 1357.558 \end{array}$$

(8)

$$\begin{array}{r|l} 8269.00 & 13'5'7.'5'5\\ \cline{2-2} 814530 & \quad 6.09112\\ \cline{1-1} 123700 & \\ 122179 & \\ \cline{1-1} 1521 & \\ 1357 & \\ \cline{1-1} 164 & \\ 135 & \\ \cline{1-1} 29 & \\ 27 & \\ \cline{1-1} 2 & \end{array}$$

(9)

$$\frac{4 \times 6.07 + 6.09112}{5} =$$

$$\frac{30.37112}{5} = 6.07423$$

= 6.0742 to five figures. Ans.

REMARK.—It was unnecessary to try 7 for one of the equal factors, because there was but a very little difference between the fifth factor 6.38 and one of the equal factors 6; in fact, when this difference is not greater than 2.5 units, it is unnecessary to try the next higher number for one of the equal factors. In this case the difference is 6.38 − 6 = .38, or less than 1 unit.

Since the difference, $1.3 - 1.253 = .047$, is less than one unit in the second figure, use 1.25 for one of the equal factors in finding the third approximation.

$$3.056 \div 1.25^4 = 1.25173+. \quad \frac{4 \times 1.25 + 1.25173}{5} = 1.25034+,$$

or 1.2503 to five figures. Ans.

The exact root to seven figures is 1.250347.

EXAMPLE 2.— $\sqrt[5]{3} = ?$

SOLUTION.—Trying 1.3 for one of the equal factors (see last example), $3 \div 1.3^4 = 1.050+$. $\frac{4 \times 1.3 + 1.050}{5} = 1.25$.

Using 1.25 for one of the equal factors, the fifth factor is $3 \div 1.25^4 = 1.22879+$, and the third approximation is $\frac{4 \times 1.25 + 1.22879}{5} = 1.24575+$. Since the sixth figure is 5, it will be well to recalculate the third approximation, using 1.246 for one of the equal factors. Hence, $3 \div 1.246^4 = 1.244656+$, and the third approximation is

$$\frac{4 \times 1.246 + 1.244656}{5} = 1.245731+, \text{ or } 1.2457 \text{ to five figures. Ans.}$$

The exact root to seven figures is 1.245731.

295. The fifth root is very seldom required; the most prominent case in practice arises in connection with finding the diameter of a pipe that will discharge a required amount of water, the head and length of pipe being known. It is also required in connection with certain problems in mine ventilation. The fourth root is used even less frequently than the fifth root. Roots higher than the fifth are never required.

TABLE METHOD OF EXTRACTING THE FIFTH ROOT.

296. In exactly the same way as in the case of square and cube roots, the first three significant figures of the fifth root may be found by means of the table of the powers of numbers.

EXAMPLE.— $\sqrt[5]{238.75} = ?$

SOLUTION.—Referring to the table, the first two figures are 2.9; the first difference is $243.00 - 205.11 = 37.89$; the second difference is $238.75 - 205.11 = 33.64$; $33.64 \div 37.89 = .88+$, or .9 to one figure. Therefore, $\sqrt[5]{238.75} = 2.99$ to three figures.

297. When finding the fifth root of numbers whose first period contains but one significant figure, carry the quotient obtained by dividing the second difference by the first difference to three decimal places, and if the third figure is 5 or a greater digit, increase the second figure by 1 and add these two figures of the quotient to those previously found for the third and fourth figures of the root. Then use all four figures when finding the third approximation.

EXAMPLE.— $\sqrt[5]{3} = ?$

SOLUTION.—Referring to the table, the first two figures are 1.2; the first difference is $3.7129 - 2.4883 = 1.2246$; the second differerce is $3 - 2.4883 = .5117$; $.5117 \div 1.2246 = .417+$, or .42 to two figures. Hence, assume that one of the equal factors is 1.242; the fifth factor is $3 \div 1.242^4 = 1.26076$, and the third approximation is $\frac{4 \times 1.242 + 1.26076}{5} = 1.245752$. Since the sixth figure is 5, recalculate the third approximation, using the result just found correct to four figures for one of the equal factors (see Example 2, Art. **294**). The result is 1.2457. Ans

ARITHMETIC.

(SECTION 6.)

INTRODUCTION.

The subject of ratio and proportion is one of the most useful of all the subjects that are taught in Arithmetic. The student will find frequent use for the principles treated of in the following pages, and is requested to study them with great care.

The student should carefully study the definitions, constantly referring to them from time to time as he progresses with the subject; he should note particularly those articles relating to inverse ratio and inverse proportion. The idea of inverse proportion is usually a difficult one for the student to grasp; but a careful study of Art. **330** and of the examples in Arts. **331** and **332** will make the matter clear to him.

Although some of the examples included between Arts. **311** and **334**, inclusive, may be solved by other methods than the use of proportion, all the examples included between the above articles, and those of similar nature included in the Examination Questions, must be solved by applying the principles of proportion; no other method of solution will be accepted. The student should study very carefully Arts. **309**, **310**, **324**, and **325**; they are very important, and they should be thoroughly understood.

The subject of compound proportion as treated in ordinary textbooks on Arithmetic usually proves of considerable difficulty to the student. The method here given, while not entirely new, presents the matter in a clearer light, we believe, than any other we have ever seen.

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RATIO.

298. Suppose that it is desired to compare two numbers, say 20 and 4. If we wish to know how many times larger 20 is than 4, we divide 20 by 4 and obtain 5 for the quotient; thus, $20 \div 4 = 5$. Hence, we say that 20 is 5 times as large as 4, i. e., 20 contains 5 times as many units as 4. Again, suppose we desire to know what part of 20 is 4. We then divide 4 by 20 and obtain $\frac{1}{5}$; thus, $4 \div 20 = \frac{1}{5}$, or .2. Hence, 4 is $\frac{1}{5}$ or .2 of 20. This operation of comparing two numbers is termed *finding the ratio* of the two numbers. Ratio, then, is a comparison. It is evident that the two numbers to be compared must be expressed in the same unit; in other words, the two numbers must both be abstract numbers or concrete numbers of the same kind. For example, it would be absurd to compare 20 horses with 4 birds, or 20 horses with 4. Hence, **ratio** may be defined as a comparison between two numbers of the same kind.

299. A ratio may be *expressed* in three ways; thus, if it is desired to compare 20 and 4, and express this comparison as a ratio, it may be done as follows: $20 \div 4$; $20 : 4$, or $\frac{20}{4}$. All three are read *the ratio of 20 to 4*. The ratio of 4 to 20 would be expressed thus: $4 \div 20$; $4 : 20$, or $\frac{4}{20}$. The first method of expressing a ratio, although correct, is seldom or never used; the second form is the one oftenest met with, while the third is rapidly growing in favor, and is likely to supersede the second. The third form, called the fractional form, is preferred by modern mathematicians, and possesses great advantages to students of Algebra and of higher mathematical subjects. The second form seems to be better adapted to arithmetical subjects, and is one we shall ordinarily adopt. There is still another way of expressing a ratio, though seldom or never used in the case of a simple ratio like that given above. Instead of the colon, a straight vertical line is used; thus, 20 | 4.

300. The **terms** of a ratio are the two numbers to be compared; thus, in the above ratio, 20 and 4 are the terms. When both terms are considered together, they are called a **couplet;** when considered separately, the first term is called the **antecedent,** and the second term the **consequent.** Thus, in the ratio 20 : 4, 20 and 4 form a couplet, and 20 is the antecedent, and 4 the consequent.

301. A ratio may be **direct** or **inverse.** The *direct ratio* of 20 to 4 is 20 : 4, while the *inverse ratio* of 20 to 4 is 4 : 20. The direct ratio of 4 to 20 is 4 : 20, and the inverse ratio is 20 : 4. An inverse ratio is sometimes called a **reciprocal** ratio. The **reciprocal** of a number is 1 divided by the number. Thus, the reciprocal of 17 is $\frac{1}{17}$; of $\frac{3}{8}$ is $1 \div \frac{3}{8} = \frac{8}{3}$; i.e., the reciprocal of a fraction is the fraction inverted. Hence, the inverse ratio of 20 to 4 may be expressed as 4 : 20, or as $\frac{1}{20} : \frac{1}{4}$. Both have equal values; for, $4 \div 20 = \frac{1}{5}$, and $\frac{1}{20} \div \frac{1}{4} = \frac{1}{20} \times \frac{4}{1} = \frac{1}{5}$.

302. The term **vary** implies a ratio. When we say that two numbers vary as some other two numbers, we mean that the ratio between the first two numbers is the same as the ratio between the other two numbers.

303. The **value** of a ratio is the result obtained by performing the division indicated. Thus, the value of the ratio 20:4 is 5, it is the quotient obtained by dividing the antecedent by the consequent.

304. By expressing the ratio in the fractional form, for example, the ratio of 20 to 4 as $\frac{20}{4}$, it is easy to see, from the laws of fractions, that if both terms be multiplied, or both divided by the same number, it will not alter the value of the ratio. Thus,

$$\frac{20}{4} = \frac{20 \times 5}{4 \times 5} = \frac{100}{20}; \text{ and } \frac{20}{4} = \frac{20 \div 4}{4 \div 4} = \frac{5}{1}.$$

305. It is also evident, from the laws of fractions, that multiplying the antecedent or dividing the consequent multiplies the ratio; and dividing the antecedent or multiplying the consequent divides the ratio.

306. When a ratio is expressed in words, as the ratio of 20 to 4, the first number named is always regarded as the antecedent and the second as the consequent, without regard to whether the ratio itself is direct or inverse. *When not otherwise specified, all ratios are understood to be direct.* To express an inverse ratio, the simplest way of doing it is to express it as if it were a direct ratio, with the first number named as the antecedent, and then transpose the antecedent to the place occupied by the consequent and the consequent to the place occupied by the antecedent; or if expressed in the fractional form, invert the fraction. Thus, to express the inverse ratio of 20 to 4, first write it 20 : 4, and then, transposing the terms, as 4 : 20; or as $\frac{20}{4}$, and then inverting as $\frac{4}{20}$. Or, the reciprocals of the numbers may be taken, as explained above. To **invert** a ratio is to transpose its terms.

EXAMPLES FOR PRACTICE.

307. What is the value of the ratio of

(*a*)	98 to 49?	Ans.	(*a*)	2.
(*b*)	\$45 to \$9?		(*b*)	5.
(*c*)	$6\frac{1}{4}$ to $\frac{1}{2}$?		(*c*)	$12\frac{1}{2}$.
(*d*)	3.5 to 4.5?		(*d*)	$.77\frac{7}{9}$.
(*e*)	The inverse ratio of 76 to 19?		(*e*)	$\frac{1}{4}$.
(*f*)	The inverse ratio of 49 to 98?		(*f*)	2.
(*g*)	The inverse ratio of 18 to 24?		(*g*)	$1\frac{1}{3}$.
(*h*)	The inverse ratio of 9 to 15?		(*h*)	$1\frac{2}{3}$.
(*i*)	The ratio of 10 to 3, multiplied by 3?		(*i*)	10.
(*j*)	The ratio of 35 to 49, multiplied by 7?		(*j*)	5.
(*k*)	The ratio of 18 to 64, divided by 9?		(*k*)	$\frac{1}{32}$.
(*l*)	The ratio of 14 to 28, divided by 5?		(*l*)	$\frac{1}{10}$.

---

308. Instead of expressing the value of a ratio by a single number, as above, it is customary to express it by means of another ratio in which the consequent is 1. Thus, suppose that it is desired to find the ratio of the weights of two pieces of iron, one weighing 45 pounds and the other weighing 30 pounds. The ratio of the heavier to the lighter is then 45 : 30, an inconvenient expression. Using the fractional form, we have $\frac{45}{30}$. Dividing both terms by 30, the consequent, we obtain $\frac{1\frac{1}{2}}{1}$ or $1\frac{1}{2} : 1$. This is the same result as obtained above, for $1\frac{1}{2} \div 1 = 1\frac{1}{2}$, and $45 \div 30 = 1\frac{1}{2}$.

309. A ratio may be squared, cubed, or raised to any power, or any root of it may be taken. Thus, if the ratio of two numbers is 105 : 63, and it is desired to cube this ratio, the cube may be expressed as $105^3 : 63^3$. That this is correct is readily seen; for, expressing the ratio in the fractional form, it becomes $\frac{105}{63}$, and the cube is $\left(\frac{105}{63}\right)^3 = \frac{105^3}{63^3} = 105^3 : 63^3$. Also, if it is desired to extract the cube root of the ratio $105^3 : 63^3$, it may be done by simply dividing the exponents by 3, obtaining 105 : 63. This may be proved in the same way as in the case of cubing the ratio. Thus, $105^3 : 63^3 = \left(\frac{105}{63}\right)^3$, and $\sqrt[3]{\left(\frac{105}{63}\right)^3} = \frac{105}{63} = 105 : 63$.

310. Since $\left(\frac{105}{63}\right)^3 = \left(\frac{5}{3}\right)^3$, it follows that $105^3 : 63^3 = 5^3 : 3^3$ (this expression is read: the ratio of 105 cubed to 63 cubed equals the ratio of 5 cubed to 3 cubed), it follows that the antecedent and consequent may always be multiplied or divided by the same number, irrespective of any indicated powers or roots, without altering the value of the ratio. Thus, $24^2 : 18^2 = 4^2 : 3^2$. For, performing the operations indicated by the exponents, $24^2 = 576$ and $18^2 = 324$. Hence, $576 : 324 = 1\frac{7}{9}$ or $1\frac{7}{9} : 1$. Also, $4^2 = 16$ and $3^2 = 9$; hence, $16 : 9 = 1\frac{7}{9}$ or $1\frac{7}{9} : 1$, the same result as before. Also, $24^2 : 18^2 = \frac{24^2}{18^2} = \left(\frac{24}{18}\right)^2 = \left(\frac{4}{3}\right)^2 = \frac{4^2}{3^2} = 4^2 : 3^2$.

The statement may be proved for roots in the same manner. Thus, $\sqrt[3]{24^3} : \sqrt[3]{18^3} = \sqrt[3]{4^3} : \sqrt[3]{3^3}$. For the $\sqrt[3]{24^3} = 24$ and $\sqrt[3]{18^3} = 18$; and, $24 : 18 = 1\frac{1}{3}$ or $1\frac{1}{3} : 1$. Also, $\sqrt[3]{4^3} = 4$ and $\sqrt[3]{3^3} = 3$; $4 : 3 = 1\frac{1}{3}$ or $1\frac{1}{3} : 1$.

Note.—If the numbers composing the antecedent and consequent have different exponents, or if different roots of those numbers are indicated, the operations described in Art. **310** cannot be performed. This is evident; for, consider the ratio $4^2 : 8^3$. When expressed in the fractional form, it becomes $\frac{4^2}{8^3}$, which cannot be expressed either as $\left(\frac{4}{8}\right)^2$ or as $\left(\frac{4}{8}\right)^3$, and, hence, cannot be reduced as described above.

PROPORTION.

311. **Proportion** is an equality of ratios, the equality being indicated by the double colon (::) or by the sign of equality (=). Thus, to write in the form of a proportion the two equal ratios, 8 : 4 and 6 : 3, which both have the same value 2, we may employ one of the three following forms:

$$8 : 4 :: 6 : 3 \qquad (1)$$

$$8 : 4 = 6 : 3 \qquad (2)$$

$$\frac{8}{4} = \frac{6}{3} \qquad (3)$$

312. The first form is the one most extensively used, by reason of its having been exclusively employed in all the older works on mathematics. The second and third forms are being adopted by all modern writers on mathematical subjects, and, in time, will probably entirely supersede the first form. In this paper we shall adopt the second form, unless some statement can be made clearer by using the third form.

313. A proportion may be *read* in two ways. The old way to read the above proportion was—*8 is to 4 as 6 is to 3*; the new way is—*the ratio of 8 to 4 equals the ratio of 6 to 3.* The student may read it either way, but we recommend the latter.

314. Each ratio of a proportion is termed a **couplet.** In the above proportion, 8 : 4 is a couplet, and so is 6 : 3.

315. The numbers forming the proportion are called **terms**; and they are numbered consecutively from left to right, thus:

$$\begin{array}{ccccccc} \textit{first} & & \textit{second} & & \textit{third} & & \textit{fourth} \\ 8 & : & 4 & = & 6 & : & 3 \end{array}$$

Hence, in any proportion, the ratio of the first term to the second term equals the ratio of the third term to the fourth term.

316. The first and fourth terms of a proportion are called the **extremes,** and the second and third terms, the **means.** Thus, in the foregoing proportion, 8 and 3 are the extremes and 4 and 6 are the means.

317. A **direct proportion** is one in which both couplets are direct ratios.

318. An **inverse proportion** is one which requires one of the couplets to be expressed as an inverse ratio. Thus, 8 is to 4 inversely as 3 is to 6 must be written $8 : 4 = 6 : 3$; i. e., the second ratio (couplet) must be inverted.

319. Proportion forms one of the most useful sections of Arithmetic. In our grandfathers' Arithmetics, it was called "The rule of three."

320. **Rule I.**—*In any proportion, the product of the extremes equals the product of the means.*

Thus, in the proportion,

$$17 : 51 = 14 : 42.$$

$$17 \times 42 = 51 \times 14, \text{ since both products equal } 714.$$

321. **Rule II.**—*The product of the extremes divided by either mean gives the other mean.*

Example.—What is the third term of the proportion $17 : 51 = \quad : 42$?

Solution.—Applying rule II, $17 \times 42 = 714$, and $714 \div 51 = 14$. Ans.

322. **Rule III.**—*The product of the means divided by either extreme gives the other extreme.*

Example.—What is the first term of the proportion $\quad : 51 = 14 : 42$?

Solution.—Applying rule III, $51 \times 14 = 714$, and $714 \div 42 = 17$. **Ans.**

323. When stating a proportion in which one of the terms is unknown, represent the missing term by a letter, as x. Thus, the last example would be written,

$$x : 51 = 14 : 42$$

and for the value of x we have $x = \frac{51 \times 14}{42} = 17$.

324. If the same (addition and subtraction excepted) operations be performed upon *all* of the terms of a proportion, the proportion is not thereby destroyed. In other words, if all of the terms of a proportion be (1) multiplied or (2) divided by the same number; (3) if all the terms be raised to the same power; if (4) the same root of all the terms be taken, or (5) if both couplets be inverted, the proportion still holds. We will prove these statements by a numerical example, and the student can satisfy himself by other similar ones. The fractional form will be used, as it is better suited to the purpose. Consider the proportion 8 : 4 = 6 : 3. Expressing it in the third form, it becomes $\frac{8}{4} = \frac{6}{3}$. What we are to prove is that, if any of the five operations enumerated above be performed upon all of the terms of this proportion, the first fraction will still equal the second fraction.

1. Multiplying all the terms by any number, say 7, $\frac{8 \times 7}{4 \times 7} = \frac{6 \times 7}{3 \times 7}$; or $\frac{56}{28} = \frac{42}{21}$. Now $\frac{56}{28}$ evidently equals $\frac{42}{21}$, since the value of either ratio is 2, and the same is true of the original proportion.

2. Dividing all the terms by any number, say 7, $\frac{8 \div 7}{4 \div 7} = \frac{6 \div 7}{3 \div 7}$; or $\frac{\frac{8}{7}}{\frac{4}{7}} = \frac{\frac{6}{7}}{\frac{3}{7}}$. But $\frac{8}{7} \div \frac{4}{7} = 2$, and $\frac{6}{7} \div \frac{3}{7} = 2$ also, the same as in the original proportion.

3. Raising all the terms to the same power, say the cube, $\frac{8^3}{4^3} = \frac{6^3}{3^3}$. This is evidently true, since $\frac{8^3}{4^3} = \left(\frac{8}{4}\right)^3 = 2^3 = 8$, and $\frac{6^3}{3^3} = \left(\frac{6}{3}\right)^3 = 2^3 = 8$ also.

4. Extracting the same root of all the terms, say the cube root, $\frac{\sqrt[3]{8}}{\sqrt[3]{4}} = \frac{\sqrt[3]{6}}{\sqrt[3]{3}}$. It is evident that this is likewise true, since $\frac{\sqrt[3]{8}}{\sqrt[3]{4}} = \sqrt[3]{\frac{8}{4}} = \sqrt[3]{2}$, and $\frac{\sqrt[3]{6}}{\sqrt[3]{3}} = \sqrt[3]{\frac{6}{3}} = \sqrt[3]{2}$ also.

5. Inverting both couplets, $\frac{4}{8}=\frac{3}{6}$, which is true, since both equal $\frac{1}{2}$.

325. If both terms of either couplet be multiplied or both divided by the same number, the proportion is not destroyed. This should be evident from the preceding article, and also from Art. **304.** Hence, in any proportion, equal factors may be canceled from the terms of a couplet, before applying rules II or III. Thus, the proportion $45 : 9 = x : 7.1$, we may divide both terms of the first couplet by 9 (that is, cancel 9 from both terms), obtaining $5 : 1 = x : 7.1$, whence $x = 7.1 \times 5 \div 1 = 35.5$. (See note in Art. **310.**)

326. The principle of all calculations in proportion is this: *Three of the terms are always given, and the remaining one is to be found.*

327. Example.—If 4 men can earn $25 in one week, how much can 12 men earn in the same time?

Solution.—The required term must bear the same relation to the given term of the same kind as one of the remaining terms bears to the other remaining term. We can then form a proportion by which the required term may be found.

The first question the student must ask himself in every calculation by proportion is:

"What is it I want to find?"

In this case it is dollars. We have two sets of men, one set earning $25, and we want to know how many dollars the other set earns. It is evident that the *amount* 12 men earn bears the same relation to the *amount* that 4 men earn as 12 men bears to 4 men. Hence, we have the proportion, the amount 12 men earn is to $25 as 12 men is to 4 men; or, since either extreme equals the product of the means divided by the other extreme, we have

$$\text{The amount 12 men earn} : \$25 = 12 \text{ men} : 4 \text{ men},$$

or the amount 12 men earn $= \frac{\$25 \times 12}{4} = \$75$. Ans.

Since it matters not which place x or the required term occupies, the problem could be stated as any of the following forms, the value of x being the same in each:

(*a*) \$25 : the amount 12 men earn = 4 men : 12 men; or the amount 12 men earn $= \frac{\$25 \times 12}{4}$, or \$75, since either mean equals the product of the extremes divided by the other mean.

(*b*) 4 men : 12 men = \$25 : the amount 12 men earn; or the amount that 12 men earn $= \frac{\$25 \times 12}{4}$, or \$75, since either extreme equals the product of the means divided by the other extreme.

(*c*) 12 men : 4 men = the amount 12 men earn : \$25; or the amount that 12 men earn $= \frac{\$25 \times 12}{4}$, or \$75, since either mean equals the product of the extremes divided by the other mean.

328. If the proportion is an inverse one, first form it as though it were a direct proportion, and then invert one of the couplets.

EXAMPLES FOR PRACTICE.

329. Find the value of x in each of the following:

(*a*) \$16 : \$64 :: x : \$4.	Ans.	(*a*) $x = \$1$.
(*b*) x : 85 :: 10 : 17.		(*b*) $x = 50$.
(*c*) 24 : x :: 15 : 40.		(*c*) $x = 64$.
(*d*) 18 : 94 :: 2 : x.		(*d*) $x = 10\frac{4}{9}$.
(*e*) \$75 : \$100 = x : 100.		(*e*) $x = 75$.
(*f*) 15 pwt. : x = 21 : 10.		(*f*) $x = 7\frac{1}{7}$ pwt.
(*g*) x : 75 yd. = \$15 : \$5.		(*g*) $x = 225$ yd.

1. If 75 pounds of lead cost \$2.10, what would 125 pounds cost at the same rate? Ans. \$3.50.

2. If A does a piece of work in 4 days and B does it in 7 days, how long will it take A to do what B does in 63 days? Ans. 36 days.

3. The circumferences of any two circles are to each other as their diameters. If the circumference of a circle 7 inches in diameter is 22 inches, what will be the circumference of a circle 31 inches in diameter? Ans. $97\frac{3}{7}$ inches.

INVERSE PROPORTION.

330. In Art. **318**, an inverse proportion was defined as one which required one of the couplets to be expressed as an inverse ratio. Sometimes the word *inverse* occurs in the

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statement of the example; in such cases the proportion can be written directly, merely inverting one of the couplets. But it frequently happens that only by carefully studying the conditions of the example can it be ascertained whether the proportion is direct or inverse. When in doubt, the student can always satisfy himself as to whether the proportion is direct or inverse by first ascertaining what is required, and stating the proportion as a direct proportion. Then, in order that the proportion may be true, if the first term is smaller than the second term, the third term must be smaller than the fourth; or if the first term is larger than the second term, the third term must be larger than the fourth term. Keeping this in mind, the student can always tell whether the required term will be larger or smaller than the other term of the couplet to which the required term belongs. Having determined this, the student then refers to the example, and ascertains from its conditions whether the required term is to be larger or smaller than the other term of the same kind. If the two determinations agree, the proportion is direct; otherwise, it is inverse, and one of the couplets must be inverted.

331. EXAMPLE.—If A's *rate* of doing work is to B's as 5 : 7, and A does a piece of work in 42 days, in what time will B do it?

SOLUTION.—The required term is the number of days it will take B to do the work. Hence, stating as a direct proportion,

$$5 : 7 = 42 : x.$$

Now, since 7 is greater than 5, x will be greater than 42. But, referring to the statement of the example, it is easy to see that B works faster than A; hence it will take B a less number of days to do the work than A. Therefore, the proportion is an inverse one, and should be stated

$$5 : 7 = x : 42,$$

from which $x = \frac{5 \times 42}{7} = 30$ days. Ans.

Had the example been stated thus: The time that A requires to do a piece of work is to the time that B requires, as 5 : 7; A can do it in 42 days, in what time can B do it? it is evident that it would take B a longer time to do the work than it would A; hence, x would be greater than 42, and the proportion would be direct, the value of x being $\frac{7 \times 42}{5} = 58.8$ days.

EXAMPLES FOR PRACTICE.

332. Solve the following:

1. If a pump which discharges 4 gal. of water per min. can fill a tank in 20 hr., how long will it take a pump discharging 12 gal. per min. to fill it? Ans. $6\frac{2}{3}$ hr.

2. If a pump discharges 90 gal. of water in 20 hr., in what time will it discharge 144 gal.? Ans. 32 hr.

3. The weight of any gas (the volume and pressure remaining the same) varies inversely as the absolute temperature. If a certain quantity of some gas weighs 2.927 lb. when the absolute temperature is 525°, what will the same volume of gas weigh when the absolute temperature is 600°, the pressure remaining the same? Ans. 2.561+ lb.

4. If 50 cu. ft. of air weigh 4.2 pounds when the absolute temperature is 562°, what will be the absolute temperature when the same volume weighs 5.8 pounds, the pressure being the same in both cases? Ans. 407°, very nearly.

POWERS AND ROOTS IN PROPORTION.

333. It was stated in Art. **309** that a ratio could be raised to any power or any root of it might be taken. A proportion is frequently stated in such a manner that one of the couplets must be raised to some power or some root of it must be taken. In all such cases, both terms of the couplet so affected *must be raised to the same power or the same root of both terms must be taken.*

334. Example.—Knowing that the weight of a sphere varies as the cube of its diameter, what is the weight of a sphere 6 inches in diameter if a sphere 8 inches in diameter of the same material weighs 180 pounds?

Solution.—This is evidently a direct proportion. Hence, we write

$$6^3 : 8^3 = x : 180.$$

Dividing both terms of the first couplet by 2^3 (see Art. **310**),

$$3^3 : 4^3 = x : 180, \text{ or } 27 : 64 = x : 180;$$

whence, $x = \frac{27 \times 180}{64} = 75\frac{15}{16}$ pounds. Ans.

Example.—A sphere 8 inches in diameter weighs 180 pounds; what is the diameter of another sphere of the same material which weighs $75\frac{15}{16}$ pounds?

Solution.—Since the weights of any two spheres are to each other as the cubes of their diameters, we have the proportion

$$180 : 75\tfrac{15}{16} = 8^3 : x^3;$$

x, the required term, must be cubed, because the other term of the couplet is cubed (see Art. **333**). But, $8^3 = 512$; hence,

$$180 : 75\tfrac{15}{16} = 512 : x^3, \text{ or } x^3 = \frac{75\frac{15}{16} \times 512}{180} = 216;$$

whence, $x = \sqrt[3]{216} = 6$ inches. Ans.

335. Since taking the same root of all of the terms of a proportion does not change its value (Art. **324**), the above example might have been solved by extracting the cube root of all of the numbers, thus obtaining $\sqrt[3]{180} : \sqrt[3]{75\frac{15}{16}} = 8 : x$; whence,

$$x = \frac{8 \times \sqrt[3]{75\frac{15}{16}}}{\sqrt[3]{180}} = 8 \times \sqrt[3]{\frac{75\frac{15}{16}}{180}} = 8\sqrt[3]{\frac{1,215}{2,880}} = 8\sqrt[3]{\frac{27}{64}} =$$

$8 \times \frac{3}{4} = 6$ inches. The process, however, is longer and is not so direct, and the first method is to be preferred.

336. If two cylinders have *equal* volumes, but different diameters, the diameters are to each other inversely as the square roots of their lengths. Hence, if it is desired to find the diameter of a cylinder that is to be 15 inches long, and which shall have the same volume as one that is 9 inches in diameter and 12 inches long, we write the proportion

$$9 : x = \sqrt{15} : \sqrt{12}.$$

Since neither 12 nor 15 are perfect squares, we square all of the terms (Arts. **335** and **324**) and obtain

$$81 : x^2 = 15 : 12; \text{ whence } x^2 = \frac{81 \times 12}{15} = 64.8,$$

and $x = \sqrt{64.8} = 8.05$ inches = diameter of 15-inch cylinder.

EXAMPLES FOR PRACTICE.

337. Solve the following examples:

1. The intensity of light varies inversely as the square of the distance from the source of light. If a gas jet illuminates an object 30 feet away with a certain distinctness, how much brighter will the object be at a distance of 20 feet? Ans. $2\frac{1}{4}$ times as bright.

2. In the last example, suppose that the object had been 40 feet from the gas jet; how bright would it have been compared with its brightness at 30 feet from the gas jet? Ans. $\frac{9}{16}$ as bright.

3. When comparing one light with another, the intensities of their illuminating powers vary as the squares of their distances from the

source. If a man can just distinguish the time indicated by his watch, 50 feet from a certain light, at what distance could he distinguish the time from a light 3 times as powerful? Ans. 86.6+ feet.

4. The quantity of air flowing through a mine varies directly as the square root of the pressure. If 60,000 cubic feet of air flow per minute when the pressure is 2.8 pounds per square foot, how much will flow when the pressure is 3.6 pounds per square foot?

Ans. 68,034 cu. ft. per min., nearly.

5. In the last example, suppose that 70,000 cubic feet per minute had been required; what would be the pressure necessary for this quantity? Ans. 3.81+ lb. per sq. ft.

CAUSES AND EFFECTS.

338. Many examples in proportion may be more easily solved by using the principle of *cause and effect.* That which may be regarded as producing a change or alteration in something, or as accomplishing something, may be called **a cause,** and the change or alteration, or thing accomplished, is the **effect.**

339. *Like causes produce like effects.* Hence, when two causes of the same kind produce two effects of the same kind, the ratio of the causes equals the ratio of the effects; in other words, the first cause is to the second cause as the first effect is to the second effect. Thus, in the question, if 3 men can lift 1,400 pounds, how many pounds can 7 men lift? we call 3 men and 7 men the *causes* (since they accomplish something, viz., the lifting of the weight), the number of pounds lifted, viz., 1,400 pounds and x pounds, are the effects. If we call 3 men the first cause, 1,400 pounds is the first effect; 7 men is the second cause, and x pounds is the second effect. Hence, we may write

$$\begin{array}{ccccccc} \textit{1st cause} & & \textit{2d cause} & & \textit{1st effect} & & \textit{2d effect} \\ 3 & : & 7 & = & 1{,}400 & : & x, \end{array}$$

whence $x = \dfrac{7 \times 1{,}400}{3} = 3{,}266\tfrac{2}{3}$ pounds.

340. The principle of cause and effect is extremely useful in the solution of examples in compound proportion, as we shall now show.

COMPOUND PROPORTION.

341. All the cases of proportion so far considered have been cases of **simple proportion**; i. e., each term has been composed of but one number. There are many cases, however, in which two or all of the terms have more than one number in them; all such cases belong to **compound proportion.** In all examples in compound proportion, both causes or both effects or all four consist of more than two numbers. We will illustrate this by an

Example.—If 40 men earn $1,280 in 16 days, how much will 36 men earn in 31 days?

Solution.—Since 40 men earn something, 40 men is a cause, and since they take 16 days in which to earn something, 16 days is also a cause. For the same reason, 36 men and 31 days are also causes. The effects, that which is earned, are 1,280 dollars and x dollars. Then, 40 men and 16 days make up the first cause, and 36 men and 31 days make up the second cause. $1,280 is the first effect and $$x$ is the second effect. Hence, we write

$$\begin{array}{ccccccc} \textit{1st cause} & & \textit{2d cause} & & \textit{1st effect} & & \textit{2d effect} \\ \begin{array}{c} 40 \\ 16 \end{array} & : & \begin{array}{c} 36 \\ 31 \end{array} & = & 1{,}280 & : & x \end{array}$$

Now, instead of using the colon to express the ratio, we shall use the vertical line (see Art. 299), and the above becomes

$$\begin{array}{c|c} 40 & 36 \\ 16 & 31 \end{array} = 1{,}280 \;\Big|\; x.$$

In the last expression, the product of all of the numbers included between the vertical lines must equal the product of all the numbers without them; i. e., $36 \times 31 \times 1{,}280 = 40 \times 16 \times x$.

$$\text{Or } x = \frac{36 \times 31 \times \overset{\overset{2}{\cancel{80}}}{\cancel{1280}}}{\cancel{40} \times \cancel{16}} = \$2{,}232. \text{ Ans.}$$

342. The above might have been solved by canceling factors of the numbers in the original proportion. For if any number within the lines has a factor common to any number without the lines, that factor may be canceled from both numbers. Thus, 16 is contained in

$$\begin{array}{c|ccc|c} & & & 2 & \\ \cancel{40} & 36 & & \cancel{80} & \\ \cancel{16} & 31 & = & \cancel{1280} & x, \end{array}$$

1,280, 80 times. Cancel 16 and 1,280, and write 80 above 1,280. 40 is contained in 80, 2 times. Cancel 40 and 80, and write 2 above 80. Now, since there are no more numbers that can be canceled, $x = 36 \times 31 \times 2 = \$2,232$, the same result as was obtained in the last article.

343. **Rule.**—*Write all the numbers forming the first cause in a vertical column, and draw a vertical line; on the other side of this line write in a vertical column all of the numbers forming the second cause. Write the sign of equality to the right of the second column, and on the right of this form a third column of the numbers composing the first effect, drawing a vertical line to the right; on the other side of this line, write, for a fourth column, the numbers composing the second effect. There must be as many numbers in the second cause as in the first cause, and in the second effect as in the first effect; hence, if any term is wanting, write x in its place. Multiply together all of the numbers within the vertical lines, and also all those without the lines (canceling previously, if possible), and divide the product of those numbers which do not contain x by the product of the others in which x occurs, and the result will be the value of x.*

344. EXAMPLE.—If 40 men can dig a ditch 720 feet long, 5 feet wide and 4 feet deep in a certain time, how long a ditch 6 feet deep and 8 feet wide could 24 men dig in the same time?

SOLUTION.—Here 40 men and 24 men are the causes and the two ditches are the effects. Hence,

$$\begin{array}{c|ccc|l} & & & \cancel{8} & \\ & & & \cancel{18} & \\ & & & \cancel{720} & x \\ \cancel{40} & 24 & = & 5 & \cancel{3} \text{ whence, } x = 24 \times 5 \times 4 = 480 \text{ feet. Ans.} \\ & & & 4 & \cancel{6} \end{array}$$

345. EXAMPLE.—The volume of a cylinder varies directly as its length and directly as the square of its diameter. If the volume of a cylinder 10 inches in diameter and 20 inches long is 1,570.8 cubic inches, what is the volume of another cylinder 16 inches in diameter and 24 inches long?

SOLUTION.—In this example, either the dimensions or the volumes may be considered the causes; say we take the dimensions for the causes. Then, squaring the diameters,

$$\begin{array}{c|cc|l} 10^2 & 16^2 & = 1,570.8 & x, \\ 20 & 24 & & \end{array} \quad \text{or} \quad \begin{array}{c|cc|l} 100 & 256 & = 1,570.8 & x; \\ \cancel{20} & \cancel{24} & & \\ 5 & 6 & & \end{array}$$

$$\text{whence, } x = \frac{256 \times 6 \times 1,570.8}{5 \times 100} = 4,825.4976 \text{ cubic inches. Ans.}$$

346. Example.—If a block of granite 8 ft. long, 5 ft. wide and 3 ft. thick weighs 7,200 lb., what will be the weight of a block of granite 12 ft. long, 8 ft. wide and 5 ft. thick?

Solution.—Taking the weights as the effects, we have

$$\begin{array}{c|c|l} \cancel{8} & \overset{4}{\cancel{12}} & \\ \cancel{5} & \cancel{8} = 7{,}200 & x, \text{ or } x = 4 \times 7{,}200 = 28{,}800 \text{ pounds. Ans.} \\ \cancel{3} & \cancel{5} & \end{array}$$

347. Example.—If 12 compositors in 30 days of 10 hours each set up 25 sheets of 16 pages each, 32 lines to the page, in how many days 8 hours long can 18 compositors set up, in the same type, 64 sheets of 12 pages each, 40 lines to the page?

Solution.—Here compositors, days, and hours compose the causes, and sheets, pages, and lines the effects. Hence,

$$\begin{array}{c|ccc|l} 3 & \cancel{3} & & \cancel{5} & 2 \\ \cancel{12} & \cancel{18} & & \cancel{25} & \cancel{64} \\ & & & \cancel{4} & \\ \cancel{30} & x & = & \cancel{16} & \cancel{12}, \text{ or } x = 3 \times 10 \times 2 = 60 \text{ days. Ans.} \\ \cancel{6} & & & & \cancel{4} \\ 10 & \cancel{8} & & \cancel{32} & \cancel{40} \\ & & & & \cancel{5} \end{array}$$

348. In examples stated like that in Art. **345**, should an inverse proportion occur, write the various numbers as in the preceding examples, and then transpose those numbers which are said to vary inversely from one side of the vertical line to the other side.

Example.—The centrifugal force of a revolving body varies directly as its weight, as the square of its velocity and inversely as the radius of the circle described by the center of the body. If the centrifugal force of a body weighing 15 pounds is 187 pounds when the body revolves in a circle having a radius of 12 inches, with a velocity of 20 feet per second, what will be the centrifugal force of the same body when the radius is increased to 18 inches and the speed is increased to 24 feet per second?

Solution.—Calling the centrifugal force the effect, we have,

$$\begin{array}{c|c|c} 15 & 15 & \\ 20^2 & 24^2 = 187 & x. \\ 12 & 18 & \end{array}$$

Transposing 12 and 18 (since the radii are to vary inversely) and squaring 20 and 24,

$$\begin{array}{c|c|l} \cancel{15} & \cancel{15} & \\ & 2 & \\ 25 & \cancel{36} = 187 & x, \text{ or } x = \dfrac{12 \times 2 \times 187}{25} = 179.52 \text{ pounds. Ans.} \\ \cancel{400} & \cancel{576} & \\ \cancel{18} & 12 & \end{array}$$

EXAMPLES FOR PRACTICE.

349. Solve the following by compound proportion:

1. If 12 men dig a trench 40 rods long in 24 days of 10 hours each how many rods can 16 men dig in 18 days of 9 hours each?
Ans. 36 rods.

2. If a piece of iron 7 ft. long, 4 in. wide, and 6 in. thick weighs 600 lb., how much will a piece of iron weigh that is 16 ft. long, 8 in. wide and 4 in. thick? Ans. 1,828$\frac{4}{7}$ lb.

3. If 24 men can build a wall 72 rods long, 6 feet wide, and 5 feet high in 60 days of 10 hours each, how many days will it take 32 men to build a wall 96 rods long, 4 feet wide and 8 feet high, working 8 hours a day? Ans. 80 days.

4. The horsepower of an engine varies as the mean effective pressure, as the piston speed and as the square of the diameter of the cylinder. If an engine having a cylinder 14 inches in diameter develops 112 horsepower when the mean effective pressure is 48 pounds per square inch and the piston speed is 500 feet per minute, what horsepower will another engine develop if the cylinder is 16 inches in diameter, piston speed is 600 feet per minute, and mean effective pressure is 56 pounds per square inch? Ans. 204.8 horsepower.

5. Referring to the example in Art. **345**, what will be the volume of a cylinder 20 inches in diameter and 24 inches long?
Ans. 7,539.84 cubic inches.

6. Knowing that the product of 3 × 5 × 7 × 9 is 945, what is the product of 6 × 15 × 14 × 36? Ans. 45,360.

FORMULAS.

350. The term **formula,** as used in mathematics and in technical books, may be defined as *a rule in which symbols are used instead of words;* in fact, a formula may be regarded as a shorthand method of expressing a rule. Any formula can be expressed in words, and when so expressed it becomes a rule.

Formulas are much more convenient than rules; they show at a glance all the operations that are to be performed; they do not require to be read three or four times, as is the case with most rules, to enable one to understand their meaning; they take up much less space, both in the printed book and in one's note-book, than rules; in short, whenever a rule can be expressed as a formula, the formula is to be preferred.

As the term "quantity" is a very convenient one to use, we will define it. In mathematics, the word **quantity** is applied to anything that it is desired to subject to the ordinary operations of addition, subtraction, multiplication, etc., when we do not wish to be more specific and state exactly what the thing is. Thus, we can say "two or more numbers," or "two or more quantities;" the word quantity is more general in its meaning than the word number.

351. The signs used in formulas are the ordinary signs indicative of operations, and the signs of aggregation. All these signs are explained in arithmetic, but some of them will here be explained in order to refresh the student's memory.

The signs indicative of operations are six in number, viz.: $+$, $-$, $\times$, $\div$, $|$, $\sqrt{}$.

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Division is indicated by the sign $\div$, or by placing a straight line between the two quantities. Thus, $25 \mid 17$, $25 / 17$, and $\frac{25}{17}$ all indicate that 25 is to be divided by 17. When both quantities are placed on the same horizontal line, the straight line indicates that the quantity on the left is to be divided by that on the right. When one quantity is below the other, the straight line between indicates that the quantity above the line is to be divided by the one below it.

The sign ($\sqrt{\ }$) indicates that some root of the quantity to the right is to be taken; it is called the **radical sign.** To indicate what root is to be taken, a small figure, called the **index,** is placed within the sign, this being always omitted when the square root is to be indicated. Thus, $\sqrt{25}$ indicates that the square root of 25 is to be taken; $\sqrt[3]{25}$ indicates that the cube root of 25 is to be taken; etc.

The signs of aggregation are four in number; viz., $\overline{\quad}$, ( ), [ ], and { }, respectively called the **vinculum,** the **parenthesis,** the **brackets,** and the **brace;** they are used when it is desired to indicate that all the quantities included by them are to be subjected to the same operation. Thus, if we desire to indicate that the sum of 5 and 8 is to be multiplied by 7, and we do not wish to actually add 5 and 8 before indicating the multiplication, we may employ any one of the four signs of aggregation as here shown: $\overline{5+8} \times 7$, $(5+8) \times 7$, $[5+8] \times 7$, $\{5+8\} \times 7$. The vinculum is placed above those quantities which are to be treated as one quantity and subjected to the same operations.

While any one of the four signs may be used as shown above, custom has restricted their use somewhat. The vinculum is rarely used except in connection with the radical sign. Thus, instead of writing $\sqrt[3]{}(5+8)$, $\sqrt[3]{}[5+8]$, or $\sqrt[3]{}\{5+8\}$ for the cube root of 5 plus 8, all of which would be correct, the vinculum is nearly always used, $\sqrt[3]{5+8}$.

In cases where but one sign of aggregation is needed (except, of course, when a root is to be indicated), the parenthesis is always used. Hence, $(5+8) \times 7$ would be the usual way of expressing the product of 5 plus 8, and 7.

If two signs of aggregation are needed, the brackets and parenthesis are used, so as to avoid having a parenthesis within a parenthesis, the brackets being placed outside. For example, $[(20-5) \div 3] \times 9$ means that the difference between 20 and 5 is to be divided by 3, and this result multiplied by 9.

If three signs of aggregation are required, the brace, brackets, and parenthesis are used, the brace being placed outside, the brackets next, and the parenthesis inside. For example, $\{[(20-5) \div 3] \times 9 - 21\} \div 8$ means that the quotient obtained by dividing the difference between 20 and 5 by 3 is to be multiplied by 9, and that after 21 has been subtracted from the product thus obtained, the result is to be divided by 8.

Should it be necessary to use all four of the signs of aggregation, the brace would be put outside, the brackets next, the parenthesis next, and the vinculum inside. For example, $\{[(\overline{20-5} \div 3) \times 9 - 21] \div 8\} \times 12$.

As stated in arithmetic, when several quantities are connected by the various signs indicating addition, subtraction, multiplication, and division, the operation indicated by the sign of multiplication must always be performed first. Thus, $2 + 3 \times 4$ equals 14, 3 being multiplied by 4, before adding to 2. Similarly, $10 \div 2 \times 5$ equals 1, since 2×5 equals 10, and $10 \div 10$ equals 1. Hence, in the above case, if the brace were omitted, the result would be $\frac{1}{4}$, whereas, by inserting the brace, the result is 36.

Following the sign of multiplication comes the sign of division in order of importance. For example, $5 - 9 \div 3$ equals 2, 9 being divided by 3 before subtracting from 5. The signs of addition and subtraction are of equal value; that is, if several quantities are connected by plus and minus signs, the indicated operations may be performed in the order in which the quantities are placed.

There is one other sign used, which is neither a sign of aggregation nor a sign indicative of an operation to be performed; it is ($=$), and is called the sign of *equality;* it means that all on one side of it is exactly equal to all on the other side. For example, $2 = 2$, $5-3 = 2$, $5 \times (14-9) = 25$.

352. Having called particular attention to certain signs used in formulas, the formulas themselves will now be explained. First, consider the well-known rule for finding the horsepower of a steam-engine, which may be stated as follows:

Divide the continued product of the mean effective pressure in pounds per square inch, the length of the stroke in feet, the area of the piston in square inches, and the number of strokes per minute, by 33,000; the result will be the horsepower.

This is a very simple rule, and very little, if anything, will be saved by expressing it as a formula, so far as clearness is concerned. The formula, however, will occupy a great deal less space, as we shall show.

An examination of the rule will show that four quantities (viz., the mean effective pressure, the length of the stroke, the area of the piston, and the number of strokes) are multiplied together, and the result is divided by 33,000. Hence, the rule might be expressed as follows:

$$\text{Horsepower} = \frac{\text{mean effective pressure}}{\text{(in pounds per square inch)}} \times \frac{\text{stroke}}{\text{(in feet)}} \times$$

$$\frac{\text{area of piston}}{\text{(in square inches)}} \times \frac{\text{number of strokes}}{\text{(per minute)}} \div 33{,}000.$$

This expression could be shortened by representing each quantity by a single letter; thus, representing horsepower by the letter "*H*," the mean effective pressure in pounds per square inch by "*P*," the length of stroke in feet by "*L*," the area of the piston in square inches by "*A*," the number of strokes per minute by "*N*," and substituting these letters for the quantities that they represent, the above expression would reduce to

$$H = \frac{P \times L \times A \times N}{33{,}000},$$

a much simpler and shorter expression. This last expression is called a *formula*.

The formula just given shows, as we stated in the beginning, that a formula is really a shorthand method of expressing a rule. It is customary, however, to omit the sign of multiplication between two or more quantities when they are to be multiplied together, or between a number and a letter representing a quantity, it being always understood that, when two letters are adjacent with no sign between them, the quantities represented by these letters are to be multiplied. Bearing this fact in mind, the formula just given can be further simplified to

$$H = \frac{PLAN}{33{,}000}.$$

The sign of multiplication, evidently, can not be omitted between two or more numbers, as it would then be impossible to distinguish the numbers. A near approach to this, however, may be attained by placing a dot between the numbers which are to be multiplied together, and this is frequently done in works on mathematics when it is desired to economize space. In such cases it is usual to put the dot higher than the position occupied by the decimal point. Thus, $2 \cdot 3$ means the same as 2×3; $542 \cdot 749 \cdot 1{,}006$ indicates that the numbers 542, 749, and 1,006 are to be multiplied together.

It is also customary to omit the sign of multiplication in expressions similar to the following: $a \times \sqrt{b+c}$, $3 \times (b+c)$, $(b+c) \times a$, etc., writing them $a\sqrt{b+c}$, $3(b+c)$, $(b+c)a$, etc. The sign is not omitted when several quantities are included by a vinculum, and it is desired to indicate that the quantities so included are to be multiplied by another quantity. For example, $3 \times \overline{b+c}$, $\overline{b+c} \times a$, $\sqrt{b+c} \times a$, etc., are always written as here printed.

353. Before proceeding further, we will explain one other device that is used by formula makers and which is apt to puzzle one who encounters it for the first time—it is the use of what mathematicians call *primes* and *subs.*, and what printers call *superior* and *inferior* characters. As a rule, formula makers designate quantities by the initial letters of the names of the quantities. For example, they represent volume by v, pressure by p, height by h, etc. This practice is to be commended, as the letter itself serves in many cases to identify the quantity which it represents.

Some authors carry the practice a little further, and represent all quantities of the same nature by the same letter throughout the book, always having the same letter represent the same thing. Now, this practice necessitates the use of the primes and subs. above mentioned, when two quantities have the same name but represent different things. Thus, consider the word *pressure* as applied to steam, at different stages between the boiler and the condenser. First, there is *absolute* pressure, which is equal to the gauge pressure in pounds per square inch plus the pressure indicated by the barometer reading (usually assumed in practice to be 14.7 pounds per square inch, when a barometer is not at hand). If this be represented by p, how shall we represent the gauge pressure? Since the absolute pressure is always greater than the gauge pressure, suppose we decide to represent it by a capital letter, and the gauge pressure by a small (lower-case) letter. Doing so, P represents absolute pressure, and p, gauge pressure. Further, there is usually a "drop" in pressure between the boiler and the engine, so that the initial pressure, or pressure at the beginning of the stroke, is less than the pressure at the boiler. How shall we represent the initial pressure? We may do this in one of three ways and still retain the letter p or P to represent the word pressure: First, by the use of the prime mark; thus, p' or P' (read *p prime* and *P major prime*) may be considered to represent the initial gauge pressure, or the initial absolute pressure. Second, by the use of sub. figures; thus, p_1 or P_1 (read *p sub. one* and *P major sub. one*). Third, by the use of sub. letters; thus, p_i or P_i (read *p sub. i* and *P major sub. i*). In the same manner p'' (read *p second*), p_2, or p_r might be used to represent the gauge pressure at release, etc. The sub. letters have the advantage of still further identifying the quantity represented; in many instances, however, it is not convenient to use them, in which case primes and subs. are used instead. The prime notation may be continued as follows: p''', p^{iv}, p^{v}, etc.; it is inadvisable to use superior figures, for example, p^1, p^2, p^3, p^a, etc., as they are liable to be mistaken for exponents.

The main thing to be remembered by the student is that *when a formula is given in which the same letters occur several times, all like letters having the same primes or subs. represent the same quantities, while those which differ in any respect represent different quantities.* Thus, in the formula

$$t = \frac{w_1 s_1 t_1 + w_2 s_2 t_2 + w_3 s_3 t_3}{w_1 s_1 + w_2 s_2 + w_3 s_3},$$

w_1, w_2, and w_3 represent the weights of three different bodies; s_1, s_2, and s_3, their specific heats; and t_1, t_2, and t_3, their temperatures; while t represents the final temperature after the bodies have been mixed together. It should be noted that those letters having the *same* subs. refer to the same bodies. Thus, w_1, s_1, and t_1 all refer to one of the three bodies; w_2, s_2, t_2, to another body; etc.

It is very easy to apply the above formula when the values of the quantities represented by the different letters are known. All that is required is to substitute the numerical values of the letters, and then perform the indicated operations. Thus, suppose that the values of w_1, s_1, and t_1 are, respectively, 2 pounds, .0951, and 80°; of w_2, s_2, and t_2, 7.8 pounds, 1, and 80°; and of w_3, s_3, and t_3, 3¼ pounds, .1138, and 780°; then, the final temperature t is, substituting these values for their respective letters in the formula,

$$t = \frac{2 \times .0951 \times 80 + 7.8 \times 1 \times 80 + 3\tfrac{1}{4} \times .1138 \times 780}{2 \times .0951 + 7.8 \times 1 + 3\tfrac{1}{4} \times .1138} =$$

$$\frac{15.216 + 624 + 288.483}{.1902 + 7.8 + .36985} = \frac{927.699}{8.36005} = 110.97°.$$

In substituting the numerical values, the signs of multiplication are, of course, written in their proper places; all the multiplications are performed before adding, according to the rule previously given.

354. The student should now be able to apply any formula involving only algebraic expressions that he may meet with, and which does not require the use of logarithms for its solution. We will, however, call his attention to one or two other facts that he may have forgotten.

Expressions similar to $\dfrac{\frac{160}{660}}{25}$ sometimes occur, the heavy line indicating that 160 is to be divided by the quotient obtained by dividing 660 by 25. If both lines were light it would be impossible to tell whether 160 was to be divided by $\frac{660}{25}$, or whether $\frac{160}{660}$ was to be divided by 25. If this latter result were desired, the expression would be written $\dfrac{\frac{160}{660}}{25}$. In every case the heavy line indicates that all above it is to be divided by all below it.

In an expression like the following, $\dfrac{160}{7+\frac{660}{25}}$, the heavy line is not necessary, since it is impossible to mistake the operation that is required to be performed. But, since $7+\frac{660}{25}=\frac{175+660}{25}$, if we substitute $\frac{175+660}{25}$ for $7+\frac{660}{25}$, the heavy line becomes necessary in order to make the resulting expression clear. Thus,

$$\frac{160}{7+\frac{660}{25}}=\frac{160}{\frac{175+660}{25}}=\frac{160}{\frac{835}{25}}.$$

Fractional exponents are sometimes used instead of the radical sign. That is, instead of indicating the square, cube, fourth root, etc., of some quantity, as 37, by $\sqrt{37}$, $\sqrt[3]{37}$, $\sqrt[4]{37}$, etc., these roots are indicated by $37^{\frac{1}{2}}$, $37^{\frac{1}{3}}$, $37^{\frac{1}{4}}$, etc. Should the numerator of the fractional exponent be some quantity other than 1, this quantity, whatever it may be, indicates that the quantity affected by the exponent is to be raised to the power indicated by the numerator; the denominator is *always* the index of the root. Hence, instead of writing $\sqrt[3]{37^2}$ for the cube root of the square of 37, it may be written $37^{\frac{2}{3}}$, the denominator being the index of the root; in other words, $\sqrt[3]{37^2} = 37^{\frac{2}{3}}$. Likewise, $\sqrt[5]{(1 + a^2 b)^3}$ may also be written $(1 + a^2 b)^{\frac{3}{5}}$, a much simpler expression.

355. We will now give several examples showing how to apply some of the more difficult formulas that the student may encounter.

1. The area of any segment of a circle that is less than (or equal to) a semicircle is expressed by the formula

$$A = \frac{\pi r^2 E}{360} - \frac{c}{2}(r - h),$$

in which A = area of segment;
π = 3.1416;
r = radius;
E = angle obtained by drawing lines from the center to the extremities of arc of segment;
c = chord of segment;
and h = height of segment.

EXAMPLE.—What is the area of a segment whose chord is 10 inches long, angle subtended by chord is 83.46°, radius is 7.5 inches, and height of segment is 1.91 inches?

SOLUTION.—Applying the formula just given,

$$A = \frac{\pi r^2 E}{360} - \frac{c}{2}(r - h) = \frac{3.1416 \times 7.5^2 \times 83.46}{360} - \frac{10}{2}(7.5 - 1.91) =$$
$$40.968 - 27.95 = 13.018 \text{ square inches, nearly.} \quad \text{Ans.}$$

2. The area of any triangle may be found by means of the following formula, in which A = the area, and a, b, and c represent the lengths of the sides:

$$A = \frac{b}{2}\sqrt{a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2}.$$

EXAMPLE.—What is the area of a triangle whose sides are 21 feet, 46 feet, and 50 feet long?

SOLUTION.—In order to apply the formula, suppose we let a represent the side that is 21 feet long; b, the side that is 50 feet long; and c, the side that is 46 feet long. Then substituting in the formula,

$$A = \frac{b}{2}\sqrt{a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2} = \frac{50}{2}\sqrt{21^2 - \left(\frac{21^2 + 50^2 - 46^2}{2 \times 50}\right)^2}$$

$$= \frac{50}{2}\sqrt{441 - \left(\frac{441 + 2{,}500 - 2{,}116}{100}\right)^2} = 25\sqrt{441 - \left(\frac{825}{100}\right)^2}$$

$$= 25\sqrt{441 - 8.25^2} = 25\sqrt{441 - 68.0625} = 25\sqrt{372.9375}$$

$$= 25 \times 19.312 = 482.8 \text{ square feet, nearly. Ans.}$$

The operations in the above examples have been extended much farther than was necessary; it was done in order to show the student every step of the process. The last formula is perfectly general, and the same answer would have been obtained had the 50-foot side been represented by a, the 46-foot side by b, and the 21-foot side by c.

3. The Rankine-Gordon formula for determining the least load in pounds that will cause a long column to break is

$$P = \frac{SA}{1 + q\frac{l^2}{G^2}},$$

in which P = load (pressure) in pounds;

S = ultimate strength (in pounds per square inch) of the material composing the column;

A = area of cross-section of column in square inches;

q = a factor (multiplier) whose value depends upon the shape of the ends of the column and on the material composing the column;

l = length of column in inches;

and G = least radius of gyration of cross-section of column.

The values of S, q, and G^2 are given in printed tables in books in which this formula occurs.

EXAMPLE.—What is the least load that will break a hollow wrought-iron column whose outside diameter is 14 inches; inside diameter, 11 inches; length, 20 feet, and whose ends are flat?

SOLUTION.—For steel, $S = 150{,}000$, and $q = \frac{1}{25{,}000}$ for flat-ended steel columns; A, the area of the cross-section, $= .7854\,(d_1^2 - d_2^2) = .7854\,(14^2 - 11^2)$, d_1 and d_2 being the outside and inside diameters, respectively; $l = 20 \times 12 = 240$ inches; and $G^2 = \frac{d_1^2 + d_2^2}{16} = \frac{14^2 + 11^2}{16}$. Substituting these values in the formula,

$$P = \frac{SA}{1 + q\frac{l^2}{G^2}} = \frac{150{,}000 \times .7854\,(14^2 - 11^2)}{1 + \frac{1}{25{,}000} \times \frac{240^2}{\frac{14^2 + 11^2}{16}}} =$$

$$\frac{150{,}000 \times 58.905}{1 + .1163} = \frac{8{,}835{,}750}{1.1163} = 7{,}915{,}211 \text{ pounds. Ans.}$$

4. Example.—When $A = 10$, $B = 8$, $C = 5$, and $D = 4$, what is the value of E in the following:

$$(a)\quad E = \sqrt[3]{\frac{BCD}{A\left(2 + \frac{D^2}{C^2}\right)}}?. \qquad (b)\quad E = \frac{A - \frac{3}{4}D + \frac{4B^2}{A + C}}{A - \sqrt{\frac{2B^2}{A + 22}}}?$$

Solution.—(*a*) Substituting,

$$E = \sqrt[3]{\frac{8 \times 5 \times 4}{10\left(2 + \frac{4^2}{5^2}\right)}}.$$

To simplify the denominator, square the 4 and 5, add the resulting fraction to 2, and multiply by 10. Simplifying, we have,

$$E = \sqrt[3]{\frac{160}{10\left(2 + \frac{16}{25}\right)}} = \sqrt[3]{\frac{160}{10 \times \frac{66}{25}}} = \sqrt[3]{\frac{160}{\frac{660}{25}}} = \sqrt[3]{\frac{200}{33}}.$$

Reducing the fraction to a decimal, so that it will be easier to extract the cube root,

$$E = \sqrt[3]{6.0606} = 1.823. \text{ Ans.}$$

(*b*) Substituting,

$$E = \frac{10 - \frac{3}{4} \times 4 + \frac{4 \times 8^2}{10 + 5}}{10 - \sqrt{\frac{2 \times 8^2}{10 + 22}}} = \frac{10 - 3 + \frac{4 \times 64}{15}}{10 - \sqrt{\frac{2 \times 64}{32}}} =$$

$$\frac{7 + 17.066+}{10 - \sqrt{4}} = \frac{24.066+}{8} = 3.008+. \text{ Ans.}$$

EXAMPLES FOR PRACTICE.

Find the numerical values of x in the following formulas, when $A = 9$, $B = 8$, $d = 10$, $e = 3$, and $c = 2$.

1. $x = \frac{d + c^2}{d^2 - 40}$. Ans. $x = \frac{7}{30}$.

2. $x = \frac{\frac{2}{3}(A + e)}{ce}$. Ans. $x = 1\frac{1}{3}$.

3. $x = \sqrt{\frac{d^2}{2c}} + \sqrt{AB^2}.$ Ans. $x = 29.$

4. $x = \frac{Ae}{\sqrt{16Bc}} + \frac{5}{16}.$ Ans. $x = 2.$

5. $x = (c + 2e)\left(\sqrt[3]{B} - \frac{1}{c}\right) + \frac{e^2 - c^2}{e^2 + c^2}.$ Ans. $x = 12\frac{5}{13}.$

6. $x = \sqrt{\frac{Bcd}{A\left(2 + \frac{d^3}{e^2}\right)}}.$ Ans. $x = .396+$

GEOMETRY AND TRIGONOMETRY.

GEOMETRY.

1. Geometry is that branch of mathematics which treats of the properties of lines, angles, surfaces, and volumes.

LINES AND ANGLES.

2. A **point** indicates position only. It has neither length, breadth, nor thickness.

3. A **line** has only one dimension: length.

4. A **straight line,** Fig. 1, is one that does not change its direction throughout its whole length. A straight line is also frequently called a **right line.**

FIG. 1.

5. A **curved line,** Fig. 2, changes its direction at every point.

FIG. 2.

6. A **broken line,** Fig. 3, is one made up wholly of straight lines lying in different directions.

FIG. 3.

7. Parallel lines, Fig. 4, are those which are equally distant from each other throughout their whole length, both lines being considered indefinite in extent. When every point of a line is the same distance from another line (or surface), it is said to be *parallel to the line* (or surface).

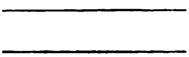

FIG. 4.

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8. A line is **perpendicular** to another when it meets that line so as not to incline towards it on either side, Fig. 5.

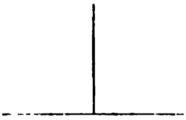


FIG. 5.

9. A **horizontal line** is a line parallel to the horizon, or water level. Fig. 6.

10. A **vertical line,** Fig. 6, is a line perpendicular to a horizontal line; consequently, it has the direction of a plumb line.

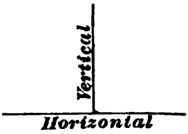


FIG. 6.

11. When two lines cross or cut each other, as in Fig. 7, they are said to **intersect,** and the point at which they intersect is called the **point of intersection,** as at A.

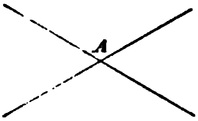


FIG. 7.

12. An **angle,** Fig. 8, is the opening between two lines that intersect or meet; the point of meeting is called the **vertex** of the angle.

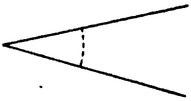


FIG. 8.

13. In order to distinguish one line from another, two of its points are given if it is a straight line, and as many more as are considered necessary if it is a broken or curved line. Thus, in Fig. 9, the line $A\,B$ would mean the straight line included between the points A and B. Similarly, the straight line between C and D would be called the line $C\,D$.

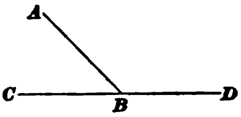


FIG. 9.

The broken line made up of the lines $A\,B$ and $B\,D$ would be called the broken line $A\,B\,D$ or $D\,B\,A$, according to the point started from. The line $C\,D$ may be regarded as a single line or as made up of two lines $C\,B$ and $B\,D$. $B\,D$ may be regarded as $C\,B$ extended, in which case it would be called *$C\,B$ produced to D*, or simply $C\,B$ **produced.** Similarly, $C\,B$ is $D\,B$ produced. One line, however, cannot be said to be another line produced, unless it is an extension of the line in a constant direction; i. e., *A B* cannot be referred to as *C B* produced or as *D B* produced.

14. To distinguish angles, name a point on each line, and the point of their intersection, or vertex of the angle. Thus, in Fig. 9, the angle formed by the lines *A B* and *C B* is called the angle *A B C* or the angle *C B A*, the letter at the vertex being placed between the other two. The angle formed by the lines *A B* and *B D* is called the angle *A B D* or the angle *D B A*.

When an angle stands alone so that it cannot be mistaken for any other angle, only the vertex letter need be given; thus, the angle *E*, Fig. 20, the angle *B*, Fig. 21, etc.

15. If one straight line meets another straight line at a point between its ends (see Figs. 9 and 10), two angles *A B C* and *A B D* are formed, which are called **adjacent angles.**

16. When adjacent angles are equal, as *A B C* and *A B D*, Fig. 10, they are called **right angles.**

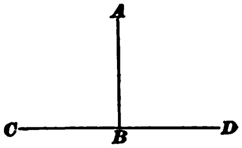


FIG. 10.

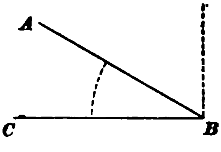


FIG. 11.

17. An **acute angle** is less than a right angle. *A B C*, Fig. 11, is an acute angle.

18. An **obtuse angle** is greater than a right angle. *A B D*, Fig. 12, is an obtuse angle.

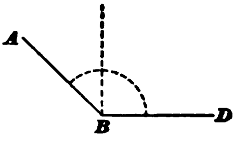


FIG. 12.

19. When two straight lines intersect they form four angles about the point of intersection. Thus, in Fig. 13, the lines $A\ B$ and $C\ D$, intersecting at the point O, form four angles $B\ O\ D$, $D\ O\ A$, $A\ O\ C$, and $C\ O\ B$ about the point O. The angles that lie on the *same* side of one straight line, as $D\ O\ B$ and $D\ O\ A$, are **adjacent angles.** The angles that lie *opposite* each other are called **opposite angles.** Thus, $A\ O\ C$ and $D\ O\ B$, also $D\ O\ A$ and $B\ O\ C$, are opposite angles.

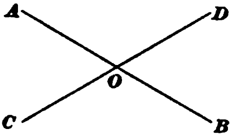


FIG. 13.

20. When one straight line intersects another straight line, as in Fig. 13, the opposite angles are equal. Thus, $D O B = A O C$, and $D O A = B O C$.

21. When one straight line meets another straight line at a point between its ends, the sum of the two adjacent angles $A\ B\ D$ and $A\ B\ C$, Fig. 14, equals two right angles.

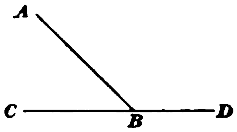


FIG. 14.

22. If a number of straight lines on the same side of a given straight line meet at the same point, the sum of all the angles formed is equal to two right angles. Thus, in Fig. 15, $C\ O\ B + D\ O\ C + E\ O\ D + F\ O\ E + A\ O\ F =$ two right angles.

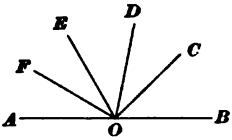


FIG 15.

23. If a straight line intersects another straight line, so that the adjacent angles are equal, the lines are said to be *perpendicular to each other*. In such a case, four right angles are formed about the point of intersection. Thus, in Fig. 16, $B\ O\ C = C\ O\ A$; hence, $B\ O\ C$, $C\ O\ A$, $A\ O\ D$, and $D\ O\ B$ are right angles. From this it is seen that *four right angles* are all that can be formed about a given point.

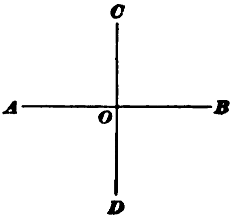


FIG. 16.

24. Through a given point any number of straight lines may be drawn; and the sum of all the angles formed about the point of intersection equals four right angles. Thus, in Fig. 17, $HOF + FOC + COA + AOG + GOE + EOD + DOB + BOH =$ four right angles.

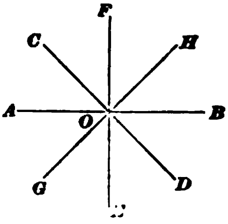


FIG. 17.

EXAMPLE.—In a flywheel with 12 arms, what part of a right angle is included between the center lines of any two adjacent arms, the arms being spaced equally?

SOLUTION.—Since there are 12 arms, there are 12 angles. The sum of all the angles equals four right angles. Hence, one angle equals $\frac{1}{12}$ of 4 right angles, or $\frac{1}{12} \times 4 = \frac{4}{12} = \frac{1}{3}$ of 1 right angle. Ans.

25. A perpendicular drawn from a point over or under a given straight line is the shortest distance from the point to the line, or to the line produced. Thus, if A, Fig. 18, is the given point and CD the given line, then the perpendicular AB is the shortest distance from A to CD.

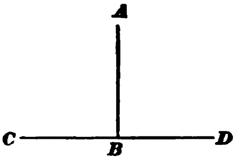


FIG. 18.

26. An angle is said to be the **complement** of another when the sum of the two angles is *one* right angle. In Fig. 17, if FE is perpendicular to AB, FOH is the complement of BOH, and BOH is the complement of FOH. When referring to both angles, they are said to be **complementary.** Thus, BOH and FOH are complementary angles.

27. When the sum of two angles is equal to *two* right angles, the angles are said to be **supplementary,** and each is the **supplement** of the other. In Fig. 14, ABC is the supplement of ABD, and ABD is the supplement of ABC. From this definition, it follows that *adjacent angles are supplementary;* also, that if one side of an angle, as BD, Fig. 14, be produced through the vertex, the angle between the side produced and the other side, i. e., the angle CBA, is the supplement of the original angle DBA.

28. If two angles have their sides parallel and both the corresponding sides lie in the same or in opposite directions, they are equal. Thus, if the side *A B*, Fig. 19, is parallel to the side *D E*, and if the side *B C* is parallel to the side *E F*, then the angle $E =$ the angle *B*. But if one of the sides of one angle lies in the same direction and the other in the opposite direction to the corresponding sides of the other angle, the angles are supplementary. Thus, in Fig. 20, *G H* is parallel to and lies in the same direction as *D E*, and *H I* is parallel to but lies in the opposite direction to *E F*; hence, angle *G H I* is the supplement of *D E F*.

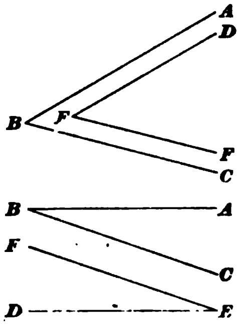


FIG. 19.

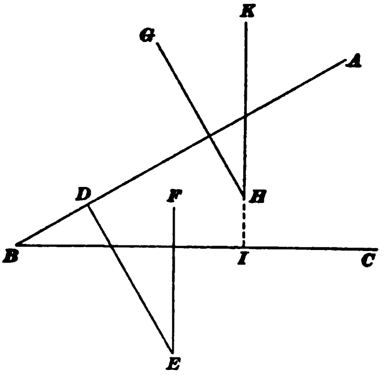


FIG. 20.

29. If two sides of an angle are perpendicular to two sides of another angle, the two angles are equal or supplementary. Thus, if *D E* and *G H*, Fig. 20, are perpendicular to *B A*, and *E F* and *H K* are perpendicular to *B C*, then will angle $E =$ angle $B =$ angle *H*; also *G H I* is the supplement of *A B C*.

EXAMPLES FOR PRACTICE.

1. In a pulley with five arms, what part of a right angle is included between the center lines of any two arms? Ans. $\frac{4}{5}$ of a right angle.

2. If one straight line meets another straight line so as to form an angle equal to $1\frac{3}{5}$ right angles, what part of a right angle does its adjacent angle equal? Ans. $\frac{2}{5}$ of a right angle.

3. If a number of straight lines meet a given straight line at a given point, all being on the same side of the given line, so as to form six equal angles, what part of a right angle is contained in each angle?

Ans. $\frac{1}{3}$ of a right angle.

PLANE FIGURES.

30. A **surface** has only two dimensions: *length* and *breadth*. A **plane surface,** usually called a **plane,** is a flat surface. If a straightedge be laid on a plane surface, every point along the edge of the straightedge will touch the surface, no matter in what direction it is laid.

31. A **plane figure** is any part of a plane surface bounded by straight or curved lines.

32. When a plane figure is bounded by **straight** lines only, it is called a **polygon.** The bounding lines are called the **sides,** and the broken line that bounds it (or the whole distance around it) is called the **perimeter** of the polygon.

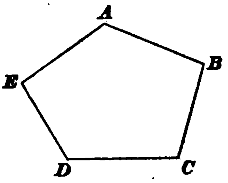


FIG. 21.

The angles formed by the sides are called the angles of the polygon. Thus, $ABCDE$, Fig. 21, is a polygon. AB, BC, etc. are the sides; EAB, ABC, etc. are the angles; and the broken line $ABCDEA$ is the perimeter.

33. Polygons are classified according to the number of their sides: One of three sides is called a **triangle;** one of four sides, a **quadrilateral;** one of five sides, a **pentagon;** one of six sides, a **hexagon;** one of seven sides, a **heptagon;** one of eight sides, an **octagon;** one of ten sides, a **decagon;** one of twelve sides, a **dodecagon;** etc.

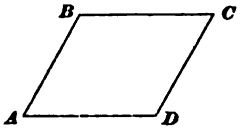


FIG. 22.

34. Equilateral polygons are those in which the sides are all equal. Thus, in Fig. 22, $AB = BC = CD = DA$; hence, $ABCD$ is an equilateral polygon.

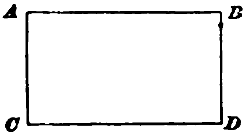


FIG. 23.

35. An **equiangular polygon** is one in which all the angles are equal. Thus, in Fig. 23, angle $A =$ angle B $=$ angle $D =$ angle C; hence, $A\,B\,D\,C$ is an equiangular polygon.

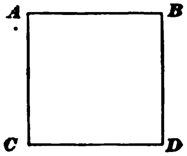
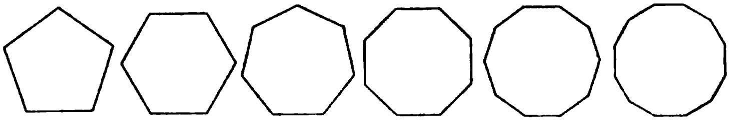


FIG. 24.

36. A **regular polygon** is one in which all the sides and all the angles are equal. Thus, in Fig. 24, $A\,B = B\,D = D\,C = C\,A$, and angle $A =$ angle B $=$ angle $D =$ angle C; hence, $A\,B\,D\,C$ is a regular polygon.

37. Some regular polygons are shown in Fig. 25.



Pentagon Hexagon Heptagon Octagon Decagon Dodecagon

FIG. 25.

38. The sum of all the interior angles of any polygon equals two right angles, multiplied by a number which is two less than the number of sides of the polygon. Thus, $A\,B\,C\,D\,E\,F$, Fig. 26, is a polygon of six sides (hexagon), and the sum of all the interior angles $A + B + C + D + E + F = 2$ right angles $\times\ 4\ (= 6 - 2)$, or 8 right angles.

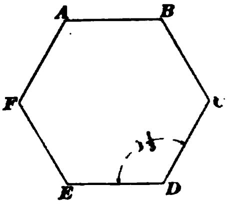


FIG. 26.

EXAMPLE.—If the above figure is a regular hexagon (has equal sides and equal angles), how many right angles are there in each interior angle?

SOLUTION.— $6 - 2 = 4$. Two right angles $\times\ 4 = 8$ right angles $=$ the total number of right angles in the polygon; and as there are 6 equal angles, we have $8 \div 6 = 1\frac{1}{3}$ right angles $=$ the number of right angles in each interior angle. Ans.

THE TRIANGLE.

39. Triangles are named according to their sides as *isosceles*, *equilateral*, and *scalene* triangles, and according to their angles as *right-angled* and *oblique-angled* triangles.

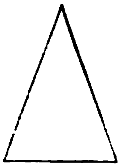


FIG. 27.

40. An **isosceles** triangle, Fig. 27, is one having two of its sides equal.

41. When the three sides are equal, as in Fig. 28, it is called an **equilateral** triangle. An equilateral triangle is also isosceles.

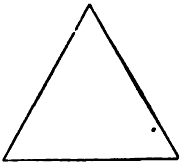


FIG. 28.

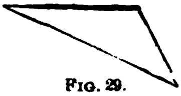


FIG. 29.

42. A **scalene** triangle, Fig. 29, is one having no two of its sides equal.

43. A **right-angled** triangle, Fig. 30, is any triangle having one right angle. The side opposite the right angle is called the **hypotenuse.** For brevity, a right-angled triangle is now termed a **right triangle.**

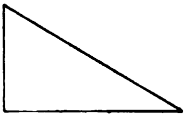


FIG. 30.

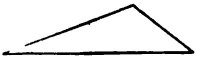


FIG. 31.

44. An **oblique** triangle, Fig. 31, is one that has no right angle.

45. The **base** of any triangle is the side upon which the triangle is supposed to stand; any side may be considered to be the base. In Figs. 32, 33, and 34, $A\,C$ is the base.

46. The **altitude** of any triangle is a line drawn from the vertex of the angle opposite the base perpendicular to the base or to the base produced. Thus, in Figs. 32 and 33, $B\,D$ is the *altitude* of the triangles $A\,B\,C$.

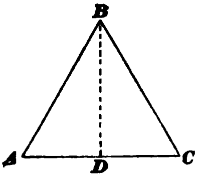


FIG. 32.

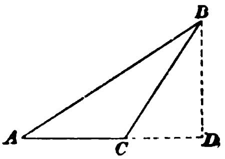


FIG. 33.

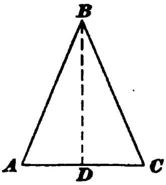


FIG. 34.

47. In an isosceles triangle, the angles opposite the equal sides are equal. Thus, in Fig. 34, $AB = BC$; hence, angle C = angle A. Therefore, if two angles of any triangle are equal, the triangle is isosceles.

In any isosceles triangle, if a perpendicular be drawn from the vertex opposite the unequal side to that side, it bisects (cuts in halves) the side. Thus, AC, Fig. 34, is the unequal side in the isosceles triangle ABC; hence, the perpendicular BD from the vertex opposite AC bisects AC, or $AD = DC$.

48. In any triangle, the sum of the three angles equals two right angles. Thus, in Fig. 35, the sum of the angles at A, B, and C = two right angles; that is, $A + B + C$ = two right angles. Hence, if any two angles of a triangle are given, the third may be found by subtracting the sum of the two from two right angles. Suppose that $A + B = 1\frac{7}{10}$ right angles; then, C must equal $2 - 1\frac{7}{10} = \frac{3}{10}$ of a right angle.

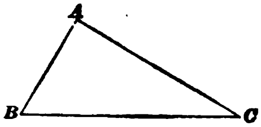


FIG. 35.

49. In any *right* triangle there can be but one right angle, and since the sum of all the angles equals two right angles, it is evident that the sum of the two acute angles must equal one right angle. Therefore, if in any right triangle one acute angle is known, the other can be found by subtracting the known angle from a right angle. Thus, in Fig. 36, ABC is a right triangle, right-angled at C. Then, the angles $A + B$ = one right angle. If $A = \frac{3}{7}$ of a right angle, $B = 1 - \frac{3}{7} = \frac{4}{7}$ of a right angle. The two acute angles of a right triangle are therefore complementary.

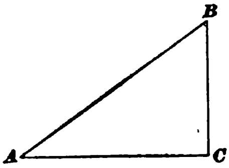


FIG. 36.

50. In any right triangle, the square described upon the hypotenuse is equal to the sum of the squares described upon the other two sides. If ABC, Fig. 37, is a right triangle, right-angled at *B*, then the square described upon the hypotenuse *A C* is equal to the sum of the squares described upon the sides *A B* and *B C*; consequently, if the lengths of the sides *A B* and *B C* are known, the length of the hypotenuse can be found by adding the squares of the lengths of the sides *A B* and *B C* and then extracting the square root of the sum.

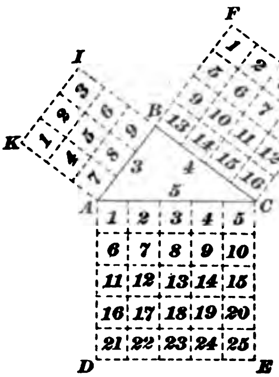


FIG. 37.

EXAMPLE.—If $A B = 3$ inches and $B C = 4$ inches, what is the length of the hypotenuse *A C*?

SOLUTION.— $3^2 = 9$; $4^2 = 16$.

Adding, $9 + 16 = 25$.

$\sqrt{25} = 5$.

Therefore, $A C = 5$ in. Ans.

51. If the hypotenuse and one side are given, the other side can be found by subtracting the square of the given side from the square of the hypotenuse, and then extracting the square root of the remainder.

EXAMPLE 1.—The side given is 3 inches, the hypotenuse is 5 inches; what is the length of the other side?

SOLUTION.— $3^2 = 9$; $5^2 = 25$. $25 - 9 = 16$, and $\sqrt{16} = 4$ in. Ans.

EXAMPLE 2.—If, from a church steeple which is 150 feet high a rope is to be attached at the top and to a stake in the ground 85 feet from its foot (the ground being supposed to be level), what must be the length of the rope?

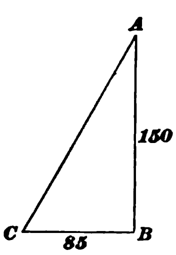


FIG. 38.

SOLUTION.—In Fig. 38, *A B* represents the steeple, 150 feet high; *C*, a stake 85 feet from the foot of the steeple; and *A C*, the rope. Here we have a right triangle, right-angled at *B*, and *A C* is the hypotenuse.

The square of $A C = 85^2 + 150^2 = 7,225 + 22,500 = 29,725$.

Therefore, $A C = \sqrt{29,725} = 172.4$ ft., nearly. Ans.

52. Two triangles are **equal** when the *sides* of one are equal to the sides of the other.

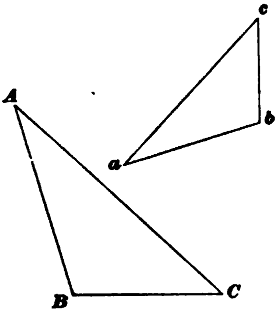


FIG. 39.

53. Two triangles are **similar** when the *angles* of one are equal to the angles of the other. *The corresponding sides of similar triangles are proportional.*

For example, in the triangles ABC and abc, Fig. 39, side ac is perpendicular to AC, side ab is perpendicular to AB, and side bc is perpendicular to BC. Hence, angle A = angle a, since the sides of one are perpendicular to the sides of the other. In like manner, angle B = angle b, and angle C = angle c. The two triangles are therefore similar and their corresponding sides are proportional. That is, any two sides of one triangle are to each other as the two corresponding sides of the other triangle; or, one side of one triangle is to the corresponding side of the other as another side of the first triangle is to the corresponding side of the second. The following are examples of the many proportions that may be written. In this case, the corresponding sides of the two triangles are the ones that are perpendicular to each other:

$$\begin{aligned} AB &: BC = ab : bc, \\ AB &: AC = ab : ac, \\ BC &: bc = AB : ab, \\ AC &: ac = BC : bc, \text{ etc.} \end{aligned}$$

EXAMPLE.—The sides of a triangle are 18 inches and 21 inches and the base is 24 inches long; what are the lengths of the sides of a similar triangle whose base is 8 inches long?

SOLUTION.—Since the sides are proportional, we have the proportions $24 : 8 = 21 : x$, and $24 : 8 = 18 : x$. From the first, $x = 7$ in., and from the second, $x = 6$ in. Ans.

54. If a straight line is drawn through two sides of a triangle parallel to the third side, it divides those sides proportionally. Thus, in Fig. 40, let the line DE be drawn parallel to the side BC in the triangle ABC. Then,

$$AD : DB = AE : EC.$$

It is to be noticed, also, that the triangles ADE and ABC are similar and their sides are proportional. The proportion $AD : DE = AB : BC$ is a useful one.

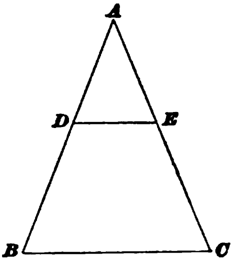


FIG. 40.

EXAMPLE 1.—In the last figure, if $AE = 14$, $AD = 12$, and $EC = 9$, what does DB equal?

SOLUTION.—From the proportion $AD : DB = AE : EC$, we have $12 : DB = 14 : 9$, whence $DB = 7\frac{5}{7}$. Ans.

EXAMPLE 2.—The base of a right triangle is 12 inches and its altitude 40 inches. How wide is the triangle 24 inches from the base?

SOLUTION.—Since the triangle is right-angled, the length of the perpendicular side equals the altitude, or 40 inches. By drawing a line parallel to the base and 24 inches above it, a second and similar triangle will be found whose corresponding side $= 40 - 24$, or 16 inches, and the length of whose base is the required width. Hence, $40 : 12 = 16 : x$, or $x = 4.8$ in. Ans.

EXAMPLES FOR PRACTICE.

1. How many right angles are there in one of the interior angles of a regular heptagon? Ans. $1\frac{3}{7}$ right angles.

2. The angle at the vertex of an isosceles triangle equals $\frac{1}{2}$ of a right angle. What do the other angles equal? Ans. $\frac{3}{4}$ of a right angle.

3. One of the acute angles of a right triangle equals $\frac{5}{8}$ of a right angle. What is the size of the other acute angle? Ans. $\frac{3}{8}$ of a right angle.

4. If the two sides about the right angle in a right triangle are 52 and 39 feet long, how long is the hypotenuse? Ans. 65 ft.

5. A ladder 65 feet long reaches to the top of a house when its foot is 25 feet from the house. How high is the house, supposing the ground to be level? Ans. 60 ft.

6. In a triangle ABC, side $AB = 32$ feet, $BC = 34$ feet, and $AC = 48$ feet. If side AB of a similar triangle is 72 feet long, what are the lengths of the other two sides? Ans. $AC = 108$ ft.; $BC = 76.5$ ft.

7. The base of a right triangle is 24 inches and its altitude 72 inches. At what distance from the top is the triangle 16 inches wide? Ans. 48 in.

THE CIRCLE.

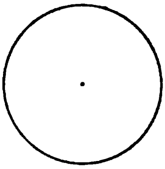


FIG. 41.

55. A **circle,** Fig. 41, is a plane figure bounded by a curved line, called the **circumference,** every point of which is equally distant from a point within, called the **center.**

56. The **diameter** of a circle *A B*, Fig. 42, is a straight line passing through the center and terminated at both ends by the circumference.

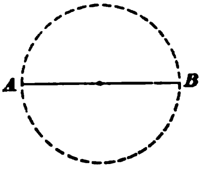


FIG. 42.

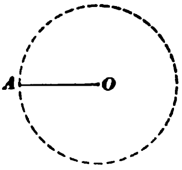


FIG. 43.

57. The **radius** of a circle, *O A*, Fig. 43, is a straight line drawn from the center to the circumference. It is equal in length to one-half the diameter. The plural of radius is **radii.** All radii of any circle are equal in length.

58. An **arc** of a circle, as *a e b*, Fig. 44, is any part of its circumference.

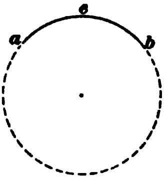


FIG. 44.

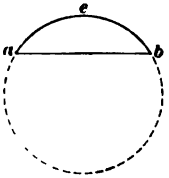


FIG. 45.

59. A **chord** is a straight line joining any two points in a circumference; or, it is a straight line joining the extremities of an arc.

Thus, in Fig. 45, *a b* is the chord of the arc *a e b*.

60. A **segment** of a circle is the space included between an arc and its chord.

Thus, in Fig. 45, the portion of the circle included between the chord *a b* and arc *a e b* is a segment.

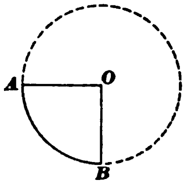


FIG. 46.

61. A **sector** of a circle is the space included between an arc and two radii drawn to the extremities of the arc.

Thus, in Fig. 46, the space included between the arc *A B* and the radii *O A* and *O B* is a sector of the circle.

62. Two circles are equal when the radius or diameter of one equals the radius or diameter of the other.

Two arcs are equal when the radius and chord of one equal the radius and chord of the other.

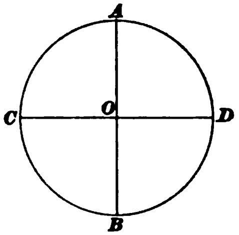


FIG. 47.

63. If *A D B C*, Fig. 47, is a circle in which two diameters *A B* and *C D* are drawn at right angles to each other, then, *A O D*, *D O B*, *B O C*, and *C O A* are right angles. The circumference is thus divided into four equal parts; each of these parts is called a **quadrant.**

64. In geometry, angles are measured by the number of right angles, or parts of a right angle, which they contain; since in the circle, a right angle intercepts a quadrant, an angle is also measured by the number of quadrants, or parts of a quadrant, that it intercepts. The word "intercept" as here used means the arc cut off by the sides of the angle.

65. An angle at the center is measured by its intercepted arc.

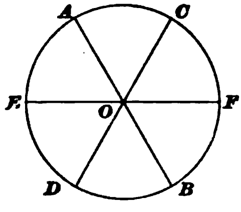


FIG. 48.

EXAMPLE.—If a circle is divided into six equal sectors, how many quadrants, or parts of a quadrant, are contained in the angle of each sector?

SOLUTION.—In Fig. 48, *A C F B D E* is a circle divided into six equal sectors. The sum of all the quadrants in the circle is 4. Hence, $4 \div 6 = \frac{2}{3}$ of a quadrant in each sector. Ans.

66. An **inscribed** angle is one whose vertex lies on the circumference of a circle and whose sides are chords. It is measured by *one-half* the intercepted arc. Thus, in Fig. 49, $A\ B\ C$ is an inscribed angle and it is measured by one-half the arc $A\ D\ C$.

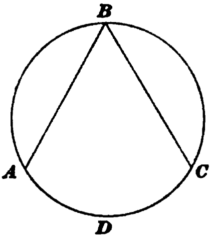


FIG 49.

EXAMPLE.—If in the figure the arc $A\ D\ C$ $= \frac{2}{5}$ of the circumference, what is the measure of the inscribed angle $A\ B\ C$?

SOLUTION.—Since the angle is an inscribed angle, it is measured by one-half the intercepted arc, or $\frac{2}{5} \times \frac{1}{2} = \frac{1}{5}$ of the circumference. The whole circumference contains four quadrants; hence, $4 \times \frac{1}{5} = \frac{4}{5}$ of a quadrant, or $\frac{4}{5}$ of a right angle. Therefore, the measure of the angle $A\ B\ C$ is $\frac{4}{5}$ of a quadrant. Ans.

67. If a circle is divided into halves, each half is called a **semicircle,** and each half circumference is called a **semi-circumference.**

68. Any angle that is inscribed in a semicircle and intercepts a semi-circumference, as $A\ B\ C$, or $A\ D\ C$, Fig. 50, is a right angle, since it is measured by one-half a semi-circumference, that is, by a quadrant.

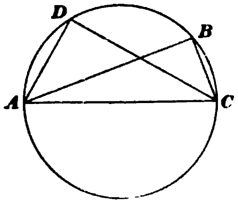


FIG. 50.

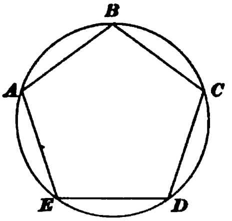


FIG. 51.

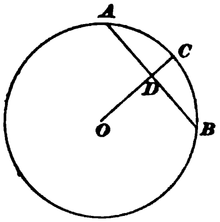


FIG. 52.

69. An **inscribed** polygon is one whose vertexes lie on the circumference of a circle, and whose sides are chords, as $A\ B\ C\ D\ E$, Fig. 51.

70. If, in any circle, a radius be drawn perpendicular to any chord, it bisects (cuts in halves) the chord. Thus, if the radius $O\ C$, Fig. 52, is perpendicular to the chord $A\ B$, $A\ D = D\ B$.

EXAMPLE.—If a regular pentagon is inscribed in a circle and a radius is drawn perpendicular to one of the sides, what are the lengths of the two parts of the side, the perimeter of the pentagon being 27 inches?

SOLUTION.—A pentagon has five sides, and since it is a regular pentagon, all the sides are of equal length; the perimeter of the pentagon, which is the distance around it, equals the sum of all the sides, or 27 inches. Therefore, the length of one side $= 27 \div 5 = 5\frac{2}{5}$ inches. Since the pentagon is an inscribed pentagon, its sides are chords, and as a radius perpendicular to a chord bisects it, we have $5\frac{2}{5} \div 2 = 2\frac{7}{10}$ inches for the length of each of the parts of the side, cut by a radius perpendicular to it. Ans.

71. If a straight line be drawn perpendicular to any chord at its middle point, it must pass through the center of the circle.

Through any three points not in the same straight line, a circumference can be drawn. Let *A*, *B*, and *C*, Fig. 53, be any three points. Join *A* and *B*, and *B* and *C*, by straight lines. At the middle point of *A B* draw *H K* perpendicular to *A B*; at the middle point of *B C* draw *E F* perpendicular to *B C*. These two perpendiculars intersect at *O*. All points on *H K* are equally distant from *A* and *B*, and all points on *E F* are equally distant from *B* and *C*; their intersection *O* is equally distant from *A*, *B*, and *C*. Then, with *O* as a center and *O B* as a radius, describe a circle; it will pass through *A*, *B*, and *C*.

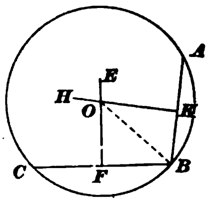


FIG. 53.

72. A **tangent** to a circle is a straight line that touches the circle at one point only; it is always perpendicular to a radius drawn to that point. Thus, in Fig. 54, *A B* drawn perpendicular to the radius *O E* at its extremity *E* is a *tangent* to the circle.

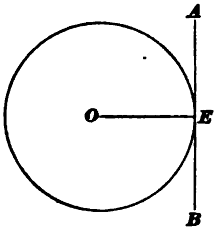


FIG. 54.

If a straight line is perpendicular to a radius at its extremity, it is tangent to the circle. Thus, in Fig. 54, if *A B* is perpendicular to the radius *O E* at *E*, *A B* is tangent to the circle.

73. If two circles intersect each other, the line joining their centers bisects at right angles the line joining the two points of intersection. If the two circles, whose centers are O and P, Fig. 55, intersect at A and B, the line OP bisects at right angles the line AB; or $AC = BC$. AB is thus perpendicular to OP.

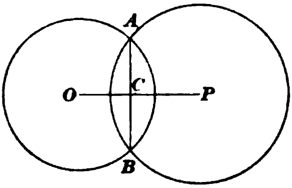


FIG. 55.

74. One circle is said to be **tangent** to another circle when they touch each other at one point only, as in Fig. 56. This point is called the **point of tangency,** or the **point of contact.**

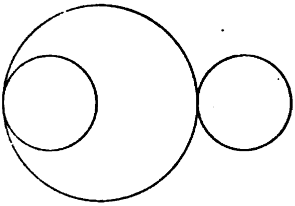


FIG. 56.

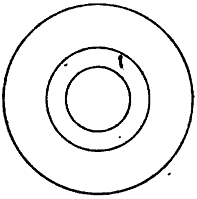


FIG. 57.

75. When two or more circles are described from the same center, as in Fig. 57, they are called **concentric** circles.

76. If, from any point on the circumference of a circle, a perpendicular be let fall upon a given diameter, this perpendicular will be a mean proportional between the two parts into which it divides the diameter.

If AB, Fig. 58, is the given diameter and C any point on the circumference, then is the perpendicular CD a mean proportional between AD and DB, or

$$AD : CD = CD : DB.$$

Therefore, $\overline{CD}^2 = AD \times DB$,

$$CD = \sqrt{AD \times DB}.$$

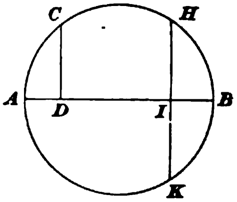


FIG. 58.

Example 1.—If $HK = 30$ feet and $IB = 8$ feet, what is the diameter of the circle, HK being perpendicular to AB?

Solution.— $30 \text{ feet} \div 2 = 15 \text{ feet} = IH.$

$$BI : IH = IH : IA, \text{ or } 8 : 15 = 15 : IA.$$

Therefore, $IA = \frac{15^2}{8} = \frac{225}{8} = 28\frac{1}{8}$ feet,

and $IA + IB = 28\frac{1}{8} + 8 = 36\frac{1}{8}$ ft. $= AB$, diameter of circle. Ans.

Example 2.—The diameter of the circle AB is $36\frac{1}{8}$ feet and the distance BI is 8 feet; what is the length of the line HK?

Solution.—As the diameter of the circle is $36\frac{1}{8}$ feet and as BI is 8 feet, IA is equal to $36\frac{1}{8} - 8 = 28\frac{1}{8}$ feet. Hence, $BI : IH = IH : IA$, or $8 : IH = IH : 28\frac{1}{8}$. Therefore, $IH = \sqrt{8 \times 28\frac{1}{8}} = 15$ feet, and as $HK = IH + IK$, or $2\,IH$, $HK = 15 \times 2 = 30$ ft. Ans.

EXAMPLES FOR PRACTICE.

1. If a circle is divided into ten equal sectors, what part of a quadrant is contained in the angle of each sector? Ans. $\frac{2}{5}$ of a quadrant.

2. An angle inscribed in a circle intercepts one-fourth of the circumference. What is the size of the angle? Ans. $\frac{1}{2}$ of a right angle.

3. The perimeter of a regular inscribed octagon is 100 inches long. If a radius is drawn perpendicular to one of the sides, what are the lengths of the two parts of the side? Ans. $6\frac{1}{4}$ in.

4. If, in Fig. 58, the diameter $AB = 32\frac{1}{2}$ feet and the distance $IB = 8$ feet, what is the length of the chord HK? Ans. 28 ft.

5. In Fig. 58, if the distance BI is 6 inches and HK 18 inches, what is the diameter of the circle? Ans. 19.5 in.

TRIGONOMETRY.

77. Trigonometry is that branch of mathematics which treats of the solution of triangles.

Every triangle has six parts—three **sides** and three **angles.** If any three of the parts are given, one of them being a side, the other three can be found. The process of finding the unknown parts from the given parts is called the **solution** of the triangle.

78. In trigonometry, the circumference of every circle is supposed to be divided into 360 equal parts, called **degrees;**

every degree is subdivided into 60 equal parts, called **minutes**; and every minute is again divided into 60 equal parts, called **seconds**. Degrees, minutes, and seconds are denoted by the symbols °, ′, ″. Thus, the expression 37° 14′ 44″, is read 37 degrees 14 minutes 44 seconds.

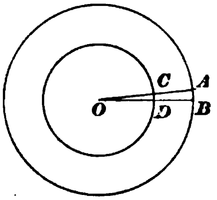


FIG. 59.

Since one degree is $\frac{1}{360}$ of any circumference, it follows that the length of an arc of one degree will be different in circles of different diameters, but the proportion of the length of an arc of one degree to the whole circumference will always be the same, viz., $\frac{1}{360}$ of the circumference.

Hence, in two given circles the length of an arc of 1° will be proportional to the two radii. Thus, if $A\,O\,B$, Fig. 59, is an angle of 1° on the larger circle, it is also 1° on the smaller concentric circle, and the length of the arc $A\,B$ is to the length of the arc $C\,D$ as the radius $O\,B$ is to the radius $O\,D$; or, $\quad$ arc $A\,B$: arc $C\,D = O\,B : O\,D$.

EXAMPLE.—If the arc $C\,D = 2$ inches, radius $O\,D = 5$ inches, and radius $O\,B = 9$ inches, what is the length of the arc $A\,B$?

SOLUTION.— $A\,B : 2 = 9 : 5$, or $A\,B = \frac{9 \times 2}{5} = 3\frac{3}{5}$ in. Ans.

79. In trigonometry, the arcs of circles are used to measure angles. All angles are supposed to have their vertexes at the center O of the circle (see Fig. 60), one side of the angle lying to the right of O, and coinciding with the horizontal diameter, as $O\,B$.

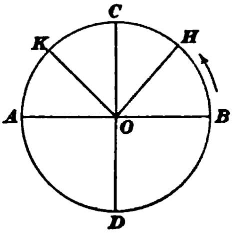


FIG. 60.

The point B on the arc is the starting point in measuring an angle; the angle is supposed to increase by moving around the circumference in the direction indicated by the arrow until the number of degrees, minutes, and seconds in the angle have been measured off on the arc. Suppose that it stops at the point H, draw $O\,H$, and $H\,O\,B$ will be the angle. If K had been the stopping point, $K\,O\,B$ would have been the angle.

In practice, angles are most conveniently laid off by using a protractor (see *Geometrical Drawing*), which is usually graduated to degrees and half degrees, minutes being estimated by the eye.

80. Since a quadrant is a fourth part of a circle, the number of degrees in a quadrant is one-fourth of 360°, or 90°. Hence, a right angle always contains 90°.

EXAMPLE.—The earth turns completely around on its axis once every day; through how many degrees does it turn in 1 hour?

SOLUTION.—In 1 day there are 24 hours, and since the earth turns through 360° in 24 hours, in 1 hour it will turn through $360° \div 24 = 15°$. Ans.

81. In adding two angles together, seconds are added to seconds, minutes to minutes, and degrees to degrees; so, also, in subtracting two angles, seconds are subtracted from seconds, minutes from minutes, and degrees from degrees.

EXAMPLE 1.—Add 75° 46′ 17″ and 14° 27′ 34″.

SOLUTION.—

$$\begin{array}{r} 75^\circ\ 46'\ 17'' \\ 14^\circ\ 27'\ 34'' \\ \hline 89^\circ\ 73'\ 51'' \end{array}$$

Since $73' = 1^\circ\ 13'$, the 1° is added to the 89°, and the sum is then written 90° 13′ 51″. Ans.

EXAMPLE 2.—What is the difference between 126° 14′ 20″ and 45° 28′ 13″?

SOLUTION.—

$$\begin{array}{r} 126^\circ\ 14'\ 20'' \\ 45^\circ\ 28'\ 13'' \\ \hline 7'' \end{array}$$

Since 28′ cannot be taken from 14′, 1° (= 60′) is taken from 126° and added to the 14′, and the above is written:

$$\begin{array}{r} 125^\circ\ 74'\ 20'' \\ 45^\circ\ 28'\ 13'' \\ \hline 80^\circ\ 46'\ \ 7''. \end{array} \text{ Ans.}$$

EXAMPLE 3.—Subtract 49° 36′ 14″ from 90°.

SOLUTION.—Since $1^\circ = 60'$ and $1' = 60''$, we can write $90^\circ = 89^\circ\ 59'\ 60''$, and

$$\begin{array}{r} 89^\circ\ 59'\ 60'' \\ 49^\circ\ 36'\ 14'' \\ \hline 40^\circ\ 23'\ 46''. \end{array} \text{ Ans.}$$

Example 4.—Add $83^\circ\ 15'\ 39''$ and $96^\circ\ 44'\ 21''$.

Solution.—

$$\begin{array}{r} 83^\circ\ 15'\ 39'' \\ 96^\circ\ 44'\ 21'' \\ \hline 179^\circ\ 59'\ 60'' \end{array}$$

Since $60'' = 1'$, add $1'$ to $59'$, making it $60'$; since $60' = 1^\circ$, add 1 to 179° making it 180°.

Therefore, $83^\circ\ 15'\ 39'' + 96^\circ\ 44'\ 21'' = 180^\circ$. Ans.

EXAMPLES FOR PRACTICE.

1. Add $43^\circ\ 0'\ 59''$ and $10^\circ\ 59'\ 40''$. Ans. $54^\circ\ 0'\ 39''$.
2. From $180^\circ\ 12'\ 20''$ subtract $3^\circ\ 12'\ 56''$. Ans. $176^\circ\ 59'\ 24''$.
3. From 84° take $83^\circ\ 14'\ 10''$, and to the result add $14'\ 10''$. Ans. 1°.

THE TRIGONOMETRIC FUNCTIONS.

82. A **function** of a quantity is another quantity depending on the first one for its value. The circumference of a circle, for example, is a function of the diameter, because the length of the circumference depends on the length of the diameter.

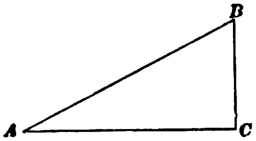


FIG. 61.

83. In the right triangle *A C B*, Fig. 61, right-angled at *C*, the size of the angle *A* (and consequently, also, of angle *B*) depends on the relative lengths of the sides *A C*, *A B*, and *B C*. No one of the sides can be changed without altering the length of at least one other side, and consequently changing the angles *A* and *B*, the angle *C* remaining a right angle. For this reason the sides are functions of the angles.

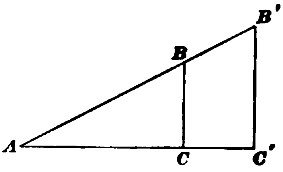


FIG. 62.

84. In Fig. 62, *A C B* is a right triangle, right-angled at *C*. The sides *A B* and *A C* have been produced to *B'* and *C'*, respectively, *B' C'* being perpendicular to *A C'* and therefore parallel to *B C*. The two triangles *A C B* and *A C' B'* are similar, since their corresponding angles are equal; hence, their corresponding sides are proportional, and we have the proportions

$$\frac{BC}{AB} = \frac{B'C'}{AB'}; \qquad \frac{BC}{AC} = \frac{B'C'}{AC'}.$$

It is evident that, no matter what the lengths of the sides of these similar triangles may be, the ratios $\frac{BC}{AB}$, $\frac{B'C'}{AB'}$, $\frac{BC}{AC}$, $\frac{B'C'}{AC'}$ will always have the same value so long as the angles remain the same. Therefore, if we knew what the values were for all angles, we could lay off any angle whatever. For, suppose that the ratio $\frac{BC}{AB}$ was known to be $\frac{1}{3}$; then, $\frac{BC}{AB} = \frac{1}{3}$ or $BC = \frac{1}{3} AB$. If we call AB, 1, then $BC = \frac{1}{3}$ and the angle can be constructed as shown in Fig. 63. Take AB as a radius and describe a circle; draw the two diameters DH and EF at right angles to each other. Lay off $AG = \frac{1}{3}$ (AB being 1), and draw GB parallel to DC, intersecting the circle in B. Then draw AB, and BAC is the required angle, since $BC = AG = \frac{1}{3} AB$.

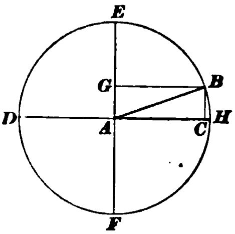


FIG. 63.

In a similar manner we can construct an angle when the ratio $\frac{BC}{AC}$ or $\frac{B'C'}{AC'}$ is known. Suppose this ratio is $\frac{3}{8}$ and that AC be taken equal to 1. With AC, Fig. 64, as a radius describe a circle and erect a perpendicular at C. Make $CB = \frac{3}{8}$ (AC being 1) and draw AB. Then, BAC is the required angle.

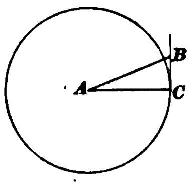


FIG. 64.

85. Suppose, in Fig. 62, the distances AC' and $B'C'$ were known, but that they were so great that it was impossible to lay them off on a drawing so that AB' could be drawn and measured; also, that it was necessary to know the direction of the line AB',

i. e., the angle A. Of course, a drawing could be made to a reduced scale; the angle A could be measured with a protractor; and the length of $A\,B'$ could be measured with a scale. The results obtained in this manner would not, in general, be accurate; the method would be long and very inconvenient, and facilities for doing this might not be at hand. If, however, we had a table giving the values of the ratio $\frac{B\,C}{A\,C}$ for all angles, we could find the value of the ratio $\frac{B'\,C'}{A\,C'}$ $\left(\text{which equals the value of the ratio } \frac{B\,C}{A\,C}\right)$, and then by looking in the table, find what angle had this value; this angle would be the angle A. The length of $A\,B'$ could be found by adding the squares of $A\,C'$ and $B'\,C'$ and extracting the square root (see Art. **50**); an easier way would be to look in a table giving the values of the ratios $\frac{B\,C}{A\,B}$ and divide $B'\,C'$ by the ratio corresponding to angle A. For representing the value of the ratio $\frac{B\,C}{A\,B}$ by R, we have

$$\frac{B\,C}{A\,B} = \frac{B'\,C'}{A\,B'} = R, \text{ or } A\,B' = \frac{B'\,C'}{R}.$$

From the foregoing, it will be perceived that the ratios mentioned are extremely important—they constitute, in fact, the foundations of trigonometry. These ratios, together with several others not yet described, are called the **trigonometric functions.**

86. There are eight trigonometric functions, the four principal ones being the *sine*, *cosine*, *tangent*, and *cotangent.* The remaining four are the *secant*, *cosecant*, *versedsine*, and *coversedsine.*

In some works on trigonometry and engineering, the trigonometric functions are treated as lines, while in others they are treated as ratios. We shall therefore define them both ways, so the student will have no difficulty in understanding either method. These functions will now be defined.

87. In any right triangle, as $O\ C\ A$, Fig. 65, right angled at C, considering the angle O, the side $A\ C$ is called the **side opposite** and the side $O\ C$ the **side adjacent**; $O\ A$ is, of course, the *hypotenuse*. Similarly, $O\ C$ is the side opposite and $A\ C$ the side adjacent for the angle A. The ratio of the side opposite to the hypotenuse is called the **sine**; that is, for the angle $A\ O\ C$,

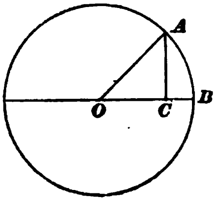


FIG. 65.

$$\text{Sine} = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{A\ C}{O\ A},$$

which is equal to $A\ C$, when $O\ A$ is taken as equal to 1. In other words, if a circle whose center is O is described *with a radius of unit length*, the perpendicular dropped from the point where one side of the angle (whose vertex is at the center of the circle) cuts the circle to the other side is the *sine*.

88. The **cosine** of an angle, as O, Fig. 65, is the ratio of the side adjacent to the hypotenuse; therefore,

$$\text{Cosine} = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{O\ C}{O\ A},$$

which is equal to $O\ C$, when the radius $O\ A = 1$. In other words, the *cosine* is the distance from the foot of the sine to the center of the circle, when the radius is unity.

89. The **tangent** of an angle, as $A\ O\ B$, Fig. 66, is the ratio of the side opposite to the side adjacent; therefore,

$$\text{Tangent} = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{D\ B}{O\ B},$$

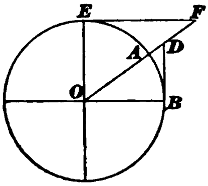


FIG. 66.

which is equal to $D\ B$, when the radius $O\ B = 1$. In other words, if a tangent is drawn at the right extremity of the horizontal diameter of a circle (described with a unit radius), which forms one side of an angle, and the other side of the angle is prolonged to meet it, the distance intercepted by the two sides of the angle is called the *tangent* of that angle.

90. The **cotangent** of an angle, as $A\,O\,B$, Fig. 66, is the ratio of the side adjacent to the side opposite; therefore,

$$\text{Cotangent} = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{O\,B}{D\,B}.$$

The cotangent is represented by the line $E\,F$, which is tangent to the circle at E, for the triangles $F\,E\,O$ and $D\,B\,O$ are similar, since they both have a right angle; the angles $E\,F\,O$ and $D\,O\,B$ are equal (see Art. **28**), and the angles $F\,O\,E$ and $O\,D\,B$ are also equal, being complements of the same angle $D\,O\,B$ (see Arts. **26** and **49**). Therefore, $\frac{O\,B}{D\,B} = \frac{E\,F}{E\,O}$. But $E\,O$ is the radius, which we assumed to be 1, and $\frac{O\,B}{D\,B}$ is the cotangent of $D\,O\,B$; hence,

$$\text{Cotangent} = \frac{O\,B}{D\,B} = \frac{E\,F}{E\,O} = EF,$$

when the radius $O\,E = 1$. In other words, if a tangent is drawn from the upper extremity of a vertical diameter of a circle, whose horizontal diameter forms one side of an angle, and the other side of the angle is produced until it meets this tangent, the distance intercepted on this tangent between the extremity of the vertical diameter and the produced line is called the *cotangent* of that angle, when the radius = 1.

91. The **secant** of an angle is the ratio of the hypotenuse to the side adjacent; therefore, referring to Fig. 67,

$$\text{Secant} = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{O\,A}{O\,C} = \frac{O\,D}{OB} = O\,D,$$

when the radius $O\,B = 1$. In other words, the *secant* is the line included between the point of intersection of the tangent with the inclined side of the angle and the center of a circle, when the radius = 1. $O\,D$ is also the secant in Fig. 66.

92. The **cosecant** is the ratio of the hypotenuse to the side opposite. Therefore, referring to Fig. 67,

$$\text{Cosecant} = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{O\,A}{A\,C}.$$

But, since $O\,R\,N$ and $O\,C\,A$ are similar right triangles, the side $O\,R$ corresponding to side $A\,C$,

$$\text{Cosecant} = \frac{OA}{OC} = \frac{O\,N}{O\,R} = O\,N,$$

when the radius $O\,B = 1$. In other words, the *cosecant* is the line included between the point of intersection of the cotangent with the inclined side of the angle and the center of a circle, when the radius $= 1$. In Fig. 66, $O\,F$ is the cosecant.

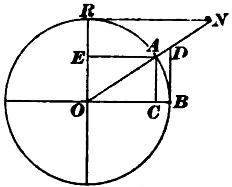


FIG. 67.

93. The versedsine and coversedsine are not generally treated as ratios. The **versedsine** is defined as 1 minus the cosine. In Fig. 67,

$$\text{Versedsine} = 1 - \text{cosine} = 1 - \frac{O\,C}{OA} = 1 - O\,C = C\,B,$$

when radius $O\,A = 1$.

The versedsine might be defined as the ratio of $C\,B$ to $O\,A$ (Fig. 67), $C\,B$ being in all cases the distance from the foot C of the sine to the right extremity B of the horizontal diameter.

The **coversedsine** is equal to 1 minus the sine. In Fig. 67, $A\,E$ is parallel to $O\,B$; hence, $E\,O = A\,C =$ sine of angle $A\,O\,C$, when radius $O\,A = 1$. Therefore,

$$\text{Coversedsine} = 1 - \text{sine} = 1 - \frac{A\,C}{OA}$$

$$= 1 - A\,C = 1 - E\,O = E\,R,$$

when radius $O\,A = 1$.

94. The four functions last defined are but little used except for special purposes; if required, they can be readily found from a table giving the values of sines, cosines, tangents, and cotangents; hence, we shall here treat only of the four functions first named.

In Art. **87**, the sine was defined as $\frac{\text{side opposite}}{\text{hypotenuse}}$, or as equal to $\frac{AC}{OA}$; in Art. **92**, the cosecant was defined as $\frac{\text{hypotenuse}}{\text{side opposite}}$, or as equal to $\frac{OA}{AC}$. It will be noticed that these two ratios are reciprocals* of each other, $\frac{AC}{OA} = \frac{1}{\frac{OA}{AC}} = \frac{AC}{OA}$. In other words, the cosecant $= \frac{1}{\text{sine}}$, and, hence, to find the cosecant of an angle, all that is necessary is to divide 1 by the sine of the angle. From this it follows that dividing by the sine is the same as multiplying by the cosecant.

Similarly, the secant is the reciprocal of the cosine; that is, secant $= \frac{1}{\text{cosine}}$. Hence, if it is required to find the secant of some angle, the secant may be found by dividing 1 by the cosine of the angle. Therefore, dividing by the cosine is equivalent to multiplying by the secant.

To find the versedsine of angle, find its cosine and subtract it from 1; to find the coversedsine, find the sine of the angle and subtract it from 1.

By comparing the ratios of the tangent and cotangent, it will be noticed that the cotangent is the reciprocal of the tangent; likewise, the tangent is the reciprocal of the cotangent.

It may be readily shown that, by dividing the ratio for the sine by that for the cosine, the tangent is equal to $\frac{\text{sine}}{\text{cosine}}$. Similarly, the cotangent is equal to $\frac{\text{cosine}}{\text{sine}}$. Hence, having given the sine and cosine of any angle, its tangent and cotangent are easily found.

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*The *reciprocal* of a number is 1 divided by the number. The reciprocal of 4 is $\frac{1}{4}$, and 4 and $\frac{1}{4}$ are said to be reciprocals of each other. The reciprocal of a fraction is the fraction inverted; thus, the reciprocal of $\frac{7}{8}$ is $\frac{8}{7}$.

95. The words cosine, cotangent, cosecant, and coversedsine are abbreviations for complement sine, complement tangent, etc., which in turn are abbreviations for the expressions "sine of complement," "tangent of complement," etc. In other words, the cosine of an angle is equal to the sine of the complement of that angle; the cotangent of an angle is equal to the tangent of its complement, etc.

That the cosine is equal to the sine of the complement is readily seen by referring to Fig. 67. Here, $A O B$ is the given angle and $A O R$ is its complement (see Art. **26**); $A C$ is its sine and $O C$ is its cosine. It is evident, from the definition of the sine, that $E A$ is the sine of the angle $A O R$. But $E A$ is equal to $O C$, since $E A C O$ is a rectangle; therefore, the cosine of $A O B$ is equal to the sine of its complement $A O R$.

Similarly, $R N$ is the tangent of $A O R$ and the cotangent of $A O B$, and $O N$ is the secant of $A O R$ and the cosecant of $A O B$. The cosine of $A O R$ is $O E$, which is equal to $A C$, the sine of $A O B$. Therefore, the versedsine of $A O R$ is $E R$, the coversedsine of $A O B$. In other words, the coversedsine of $A O B$ is equal to the versedsine of $A O R$, the complement of $A O B$.

96. In order to save time and space in writing, the names of the functions are abbreviated as follows: **sin** for sine; **cos** for cosine; **tan** for tangent; **cot** for cotangent; **sec** for secant; **csc** or **cosec** for cosecant; **vers** for versedsine; and **cvs** or **covers** for coversedsine. These abbreviations are used only when referring directly to angles; when the names are used in a general sense, they are written out in full. Let A represent some angle; then, if it were desired to refer to the sine, tangent, etc. of this angle, it would be written sin A, tan A, etc., and these expressions would be read "sine A," "tangent A," etc.

These abbreviations must always be pronounced in full. Thus, cos 14° 22′ 46″ is pronounced *cosine fourteen degrees twenty-two minutes forty-six seconds;* tan 45° is pronounced *tangent forty-five degrees.*

97. To facilitate calculations, tables of the trigonometric functions are employed. These tables give the sine, cosine, tangent, and cotangent of the degrees and minutes in a circle whose radius is 1. There are two kinds of tables giving the trigonometric functions; viz., the table of *natural functions* and the table of *logarithmic functions.* The table of **natural functions** gives the actual values of the ratios, while the table of **logarithmic functions** gives the logarithms of the natural functions. Only the table of natural functions is described in the present text.

98. From the definitions of the various trigonometric functions we derive the following very useful rules for right triangles:

$$\textit{Sine} = \frac{\textit{side opposite}}{\textit{hypotenuse}}; \text{ therefore,}$$

Rule 1.—*Side opposite = hypotennse × sine.*[*]

Rule 2.—$\textit{Hypotenuse} = \frac{\textit{side opposite}}{\textit{sine}}.$

$$\textit{Cosine} = \frac{\textit{side adjacent}}{\textit{hypotenuse}}; \text{ therefore,}$$

Rule 3.—*Side adjacent = hypotenuse × cosine.*

Rule 4.—$\textit{Hypotenuse} = \frac{\textit{side adjacent}}{\textit{cosine}}.$

$$\textit{Tangent} = \frac{\textit{side opposite}}{\textit{side adjacent}}; \text{ therefore,}$$

Rule 5.—*Side opposite = side adjacent × tangent.*

$$\textit{Cotangent} = \frac{\textit{side adjacent}}{\textit{side opposite}}; \text{ therefore,}$$

Rule 6.—*Side adjacent = side opposite × cotangent.*

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[*] Since the quotient equals the dividend divided by the divisor, the dividend equals the product of the divisor and quotient.

TRIGONOMETRIC TABLES.

99. We shall now explain how to find the sine, cosine, tangent, and cotangent of an angle by means of the table of natural trigonometric functions that accompanies this text. It may here be remarked that the values of the functions are never calculated directly (except in making a table), because the process is so long and laborious that it would require considerable time to calculate even the value of one function of a single angle, and there is no simple method of determining the angle corresponding to a given function, except by aid of a table. As they are not necessary, the secants, cosecants, versedsines, and coversedsines are omitted entirely.

100. Given an angle, to find its sine, cosine, tangent, and cotangent:

EXAMPLE 1.—Let it be required to find the sine, cosine, tangent, and cotangent of an angle of 37° 24′.

SOLUTION.—Look in the table of ***natural sines*** along the tops of the pages and find 37°. The left-hand column is marked (′), meaning that the minutes are to be sought in that column, and begin with 0, 1, 2, 3, etc., to 60. Glancing *down* this column until 24′ is found, find opposite this 24′ in the column marked *sine*, and headed 37°, the number .60738; then, $.60738 = \sin 37° 24'$. In exactly the same manner, find opposite 24′ in the column marked *cosine*, and headed 37°, the number .79441, which corresponds to cos 37° 24′; or $\cos 37° 24' = .79441$. So, also, find in the column marked *tangent*, and headed 37°, and opposite 24′, the number .76456; whence, $\tan 37° 24' = .76456$. Finally, find in the column marked *cotangent*, and headed 37°, and opposite 24′, the number 1.30795; whence, $\cot 37° 24' = 1.30795$.

In most of the tables published, the angles run only from 0° to 45°, at the heads of the columns; to find an angle greater than 45°, *look at the bottom of the page and glance upwards*, **using the extreme right-hand column to find minutes,** which begin with 0 at the *bottom* and run upwards, 1, 2, 3, etc., to 60.

EXAMPLE 2.—Find the sine, cosine, tangent. and cotangent of 77° 43′.

SOLUTION.—Since this angle is greater than 45°, look along the bottom of the tables, until the column marked ***sine*** at the bottom, and having 77° under it, is found. Glancing *up* the column of minutes on the *right*, until 43′ is found, find opposite 43′ in the column marked *sine* at the bottom, and having 77° under it, the number .97711; this is the sine of 77° 43′, or sin 77° 43′ = .97711. Similarly, in the column marked *cosine*, and having 77° under it, find opposite 43′, in the right-hand column, the number .21275; this is the cosine of 77° 43′, or cos 77° 43′ = .21275. So, also, find that 4.59283 is the tangent of 77° 43′, or tan 77° 43′ = 4.59283. Finally, in the same manner, find that the cotangent of 77° 43′ or cot 77° 43′ = .21778.

101. Let it be required to find the sine of 14° 22′ 26″.

EXPLANATION.—The sine of 14° 22′ 26″ lies between the sine of 14° 22′ and the sine of 14° 23′. For a difference of 1 minute or less between two or more angles, it is correct to assume that the differences in the values of the sine, cosine, etc. of the angles are proportional to the differences in the number of seconds in these angles. The difference in the number of seconds between 14° 22′ and 14° 22′ 26″ is 26″, and between 14° 22′ and 14° 23′ is 60″. The sine of 14° 22′ is .24813; sine of 14° 23′ is .24841. The difference between the value of the sine of 14° 22′ and the sine of 14° 22′ 26″ is not known; hence, represent it by x. The difference between the value of the sine of 14° 22′ and the sine of 14° 23′ is .24841 − .24813 = .00028, or 28 parts. Therefore, we have the proportion

$$26'' : 60'' = x \text{ parts} : 28 \text{ parts, or } \frac{26''}{60''} = \frac{x \text{ parts}}{28 \text{ parts}},$$

from which $x \text{ parts} = \frac{26}{60} \times 28 = 12.1 \text{ parts.}$

Neglecting the .1, since .1 is less than .5, we must add 12 parts, or .00012, to .24813 to obtain the sine of 14° 22′ 26″. Hence, sin 14° 22′ 26″ = .24813 + .00012 = .24825.

102. By referring to the table of sines, cosines, tangents, and cotangents, it will be observed that, as the angles increase in size, the sines and tangents increase, while the *cosines and cotangents decrease*. In the above example, therefore, had it been required to find the *cosine* or the *cotangent* of 14° 22′ 26″, the correction for the 26″ would have been *subtracted* from the cosine or the cotangent of 14° 22′ instead of added to it. The reason for this will be made apparent on referring to Fig. 67. Here it will be seen that as the sine and tangent increase, the cosine and cotangent decrease, and vice versa. From the foregoing we have, to find the sine, cosine, tangent, or cotangent of an angle containing seconds, the following rule:

Rule 7.—*Find in the table the sine, cosine, tangent, or cotangent corresponding to the degrees and minutes of the angle.*

For the seconds, find the difference between this value and the value of the sine, cosine, tangent, or cotangent of an angle 1 minute greater; multiply this difference by a fraction whose numerator is the number of seconds in the given angle and whose denominator is 60.

If the sine or tangent is sought, add this correction to the value first found; if the cosine or cotangent is sought, subtract the correction.

EXAMPLE.—Find the sine, cosine, tangent, and cotangent of 56° 43′ 17″.

SOLUTION.—Sin 56° 43′ = .83597. Sin 56° 44′ = .83613. Since 56° 43′ 17″ is greater than 56° 43′ and less than 56° 44′, the value of the sine of the angle lies between .83597 and .83613; the difference = .83613 − .83597 = .00016. Multiplying this by the fraction $\frac{17}{60}$, $.00016 \times \frac{17}{60} = .00005$, nearly, which is to be *added* to .83597, the value first found, or .83597 + .00005 = .83602. Hence, sin 56° 43′ 17″ = .83602. Ans.

Cos 56° 43′ = .54878; cos 56° 44′ = .54854; the difference = .54878 − .54854 = .00024, and $.00024 \times \frac{17}{60} = .00007$, nearly. Now, since the *cosine* is sought, we must *subtract* this correction from cos 56° 43′ or .54878; subtracting, .54878 − .00007 = .54871. Hence, cos 56° 43′ 17″ = .54871. Ans.

Tan 56° 43′ = 1.52332; tan 56° 44′ = 1.52429; the difference = .00097, and $.00097 \times \frac{17}{60} = .00027$, nearly. Since the tangent is sought, we must *add*, giving 1.52332 + .00027 = 1.52359. Hence, tan 56° 43′ 17″ = 1.52359. Ans.

Cot 56° 43′ = .65646; cot 56° 44′ = .65604; the difference = .00042, and $.00042 \times \frac{17}{60} = .00012$, nearly. Since the cotangent is sought, we must *subtract*, giving .65646 − .00012 = .65634. Hence, cot 56° 43′ 17″ = .65634. Ans.

103. Given the sine, cosine, tangent, or cotangent, to find the angle corresponding:

EXAMPLE 1.—The sine of an angle is .47486; what is the angle?

SOLUTION.—Consulting the table of natural sines, glance down the columns marked *sine* until .47486 is found opposite 21′, in the left-hand column, and under the column headed 28°. Therefore, the angle whose sine = .47486 is 28° 21′, or sin 28° 21′ = .47486. Ans.

EXAMPLE 2.—Find the angle whose cosine is .27032.

SOLUTION.—Looking in the columns marked *cosine*, at the top of the page, it is not found; hence, the angle is greater than 45°. Consequently, looking in the columns marked *cosine* at the bottom of the page, it is found opposite 19′, in the *right-hand* column of minutes, and in the column having 74° at the bottom. Therefore, the angle whose cosine is .27032 is 74° 19′, or cos 74° 19′ = .27032. Ans.

EXAMPLE 3.—Find the angle whose tangent is 2.15925.

SOLUTION.—On searching the table of natural tangents, the given tangent is found to belong to an angle greater than 45°, so it must be looked for in the column marked *tangent* at the bottom. It is found opposite 9′, in the right-hand column of minutes and in the column having 65° at the bottom. Therefore, tan 65° 9′ = 2.15925. Ans.

EXAMPLE 4.—Find the angle whose cotangent is .43412.

SOLUTION.—From the table of natural cotangents, it is found that this value is less than the cotangent of 45°, so it must be found in the column marked *cotangent* at the bottom. Looking there, it is found in the column having 66° at the bottom, and opposite 32′, in the right-hand column of minutes. Therefore, the angle whose cotangent is .43412 is 66° 32′, or cot 66° 32′ = .43412. Ans.

104. Let it be required to find the angle whose sine is .42531.

EXPLANATION.—Referring to the table of sines, this number is found to lie between .42525, the sine of 25° 10′, and .42552, the sine of 25° 11′. The difference between these two numbers is .42552 − .42525 = .00027, or 27 parts; the difference between .42525, the sine of 25° 10′, and .42531, the sine of the given angle, is .42531 − .42525 = .00006, or 6 parts. Representing by x the number of seconds that the angle whose sine is .42531 exceeds 25° 10′, we have the proportion

$$x'' : 60'' = 6 \text{ parts} : 27 \text{ parts},$$

or

$$\frac{x''}{60''} = \frac{6 \text{ parts}}{27 \text{ parts}};$$

from which $x = 60 \times \frac{6}{27} = 13.3''$. Hence, the angle whose sine is .42531 is 25° 10′ 13.3″.

The angle is found from the cosine, tangent, and cotangent in exactly the same manner.

105. To find the angle corresponding to a given sine, cosine, tangent, or cotangent, whose exact value is not contained in the table:

Rule 8.—*Find the difference of the two numbers in the table between which the given sine, cosine, tangent, or cotangent falls, and use the number of parts in this difference as the denominator of a fraction.*

Find the difference between the number belonging to the **smaller angle** *and the given sine, cosine, tangent, or cotangent, and use the number of parts in the difference just found as the numerator of the fraction mentioned above. Multiply this fraction by 60, and the result will be the number of seconds to be added to the* **smaller angle.**

EXAMPLE 1.—Find the angle whose sine is .57698.

SOLUTION.—Looking in the table of natural sines, in the columns marked *sine*, it is found between .57691 = sin 35° 14′ and .57715 = sin 35° 15′. The difference between them is .57715 − .57691 = .00024, or 24 parts. The difference between the sine of the smaller angle, or sin 35° 14′ = .57691, and the given sine, or .57698, is .57698 − .57691 = .00007, or 7 parts. Then, $\frac{7}{24} \times 60 = 17.5''$, and the required angle is 35° 14′ 17.5″, or sin 35° 14′ 17.5″ = .57698. Ans.

EXAMPLE 2.—Find the angle whose cosine is .27052.

SOLUTION.—Looking in the table of cosines, it is found to belong to a greater angle than 45° and, hence, must be sought for in the columns marked *cosine*, at the bottom of the page. It is found between the numbers .27060 = cos 74° 18′ and .27032 = cos 74° 19′. The difference between the two numbers is .27060 − .27032 = .00028, or 28 parts. The cosine of the *smaller angle*, or 74° 18′, is .27060, and the difference between this and the given cosine is .27060 − .27052 = .00008, or 8 parts. Hence, $\frac{8}{28} \times 60 = 17.1''$, nearly, and the angle whose cosine is .27052 = 74° 18′ 17.1″, or cos 74° 18′ 17.1″ = .27052. Ans.

EXAMPLE 3.—Find the angle whose tangent is 2.15841.

SOLUTION.— 2.15841 falls between 2.15760 = tan 65° 8′ and 2.15925 = tan 65° 9′. The difference between these numbers is 2.15925 − 2.15760

= .00165, or 165 parts. 2.15841 − 2.15760 = .00081, or 81 parts. Hence, $\frac{81}{165} \times 60 = 29.5''$, nearly, and the angle whose tangent is 2.15841 = 65° 8′ 29.5″, or tan 65° 8′ 29.5″ = 2.15841. Ans.

EXAMPLE 4.—Find the angle whose cotangent is 1.26342.

SOLUTION.— 1.26342 falls between 1.26395 = cot 38° 21′ and 1.26319 = cot 38° 22′. The difference between these numbers is 1.26395 − 1.26319 = .00076. 1.26395 − 1.26342 = .00053. $\frac{53}{76} \times 60 = 41.9$, nearly, and the angle whose cotangent is 1.26342 = 38° 21′ 41.9″, or cot 38° 21′ 41.9″ = 1.26342. Ans.

EXAMPLES FOR PRACTICE.

1. Find the (*a*) sine, (*b*) cosine, and (*c*) tangent of 48° 17′.

$$\text{Ans.}\left\{\begin{array}{ll}(a) & .74644.\\ (b) & .66545.\\ (c) & 1.12172.\end{array}\right.$$

2. Find the (*a*) sine, (*b*) cosine, and (*c*) tangent of 13° 11′ 6″.

$$\text{Ans.}\left\{\begin{array}{ll}(a) & .22810.\\ (b) & .97364.\\ (c) & .23427.\end{array}\right.$$

3. Find the (*a*) sine, (*b*) cosine, and (*c*) tangent of 72° 0′ 1.8″.

$$\text{Ans.}\left\{\begin{array}{ll}(a) & .95106.\\ (b) & .30901.\\ (c) & 3.07777.\end{array}\right.$$

4. (*a*) Of what angle is .26489 the sine? (*b*) Of what is it the cosine?

$$\text{Ans.}\left\{\begin{array}{ll}(a) & 15^\circ\ 21'\ 37.2''.\\ (b) & 74^\circ\ 38'\ 22.8''.\end{array}\right.$$

5. (*a*) Of what angle is .68800 the sine? (*b*) Of what the cosine? (*c*) Of what the tangent?

$$\text{Ans.}\left\{\begin{array}{ll}(a) & 43^\circ\ 28'\ 20''.\\ (b) & 46^\circ\ 31'\ 40''.\\ (c) & 34^\circ\ 31'\ 40.5''.\end{array}\right.$$

THE SOLUTION OF TRIANGLES.

RIGHT TRIANGLES.

106. As previously stated, every triangle has six parts, three sides and three angles, and if any three parts are given, one of them being a side, the other three may be found.

In right triangles, it is only necessary to know *two* parts in addition to the right angle, one of which must be a side.

Rules **1** to **8** and the definitions of sine, cosine, tangent, and cotangent are sufficient for solving all cases of right triangles. The method is best illustrated by examples. There are two cases.

107. Case I.—*When the two given parts are a side and an angle:*

Example 1.—In Fig. 68, the length of the hypotenuse *A B* of the right triangle *A C B*, right-angled at *C*, is 24 feet, and the angle *A* is 29° 31′; find the sides *A C* and *B C* and the angle *B*.

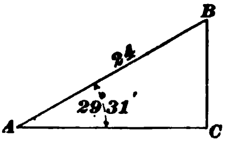

Fig. 68.

Note.—When working examples of this kind, construct the figure and mark the known parts. This is a great help in solving the example. Hence, in the figure, draw the angle *A* to represent an angle of 29° 31′ and complete the right triangle *A C B*, right-angled at *C*, as shown. Mark the angle *A* and the hypotenuse, as is done in the figure.

Solution.—Referring to Art. **49**, angle $B = 90° - 29° 31' = 60° 29'$. To find *A C*, use rule **3**; viz., *A C*, or side adjacent = hypotenuse × cosine = 24 × cos 29° 31′ = 24 × .87021 = 20.89 feet, nearly.

To find *B C*, use the same rule; thus, *B C* = 24 × cos 60° 29′ = 24′ × .49268 = 11.82 feet, nearly. To find *B C*, rule **1** could also have been used, viz., side opposite = hypotenuse × sine, or *B C* = 24 × sin 29° 31′ = 24 × .49268 = 11.82 feet, nearly.

Ans. { Angle *B* = 60° 29′.
Side *A C* = 20.89 ft.
Side *B C* = 11.82 ft.

Example 2.—One side of a right triangle *A C B*, right-angled at *C*, Fig. 69, is 37 feet 7 inches long; the angle opposite is 25° 33′ 7″. What are the lengths of the hypotenuse and the side adjacent, and what is the other angle?

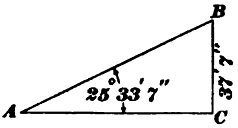

Fig. 69.

Solution.—Angle $B = 90° - 25° 33' 7'' = 64° 26' 53''$.

To find the hypotenuse, use rule **2**,

$$\text{Hypotenuse} = \frac{\text{side opposite}}{\text{sine}}.$$

Since the side opposite is given in feet and inches, both must be reduced to feet or both to inches.

7 inches = $\frac{7}{12}$ foot = .583+ foot; hence, *B C* = 37.583 feet.

Therefore, the hypotenuse is equal to

$$\frac{37.583}{\sin 25° 33' 7''} = \frac{37.583}{.43133} = 87.133 \text{ feet} = 87 \text{ feet } 2 \text{ inches, nearly.}$$

To find the side $A\,C$, use rule **3**; side adjacent = hypotenuse × cosine = 87.133 × cos 25° 33′ 7″ = 87.133 × .90219 = 78.61 feet = 78 feet 7¼ inches. nearly.

$$\text{Ans.}\begin{cases}\text{Angle } B = 64° \, 26' \, 53''.\\ A\,C = 78 \text{ ft. } 7\tfrac{1}{4} \text{ in.}\\ A\,B = 87 \text{ ft. } 2 \text{ in.}\end{cases}$$

The work involved in finding the sine and cosine of 25° 33′ 7″, in the above example, is as follows: Sin 25° 33′ = .43130; sin 25° 34′ = .43156; difference = .00026; .00026 × $\frac{7}{60}$ = .00003. Hence, sin 25° 33′ 7″ = .43130 + .00003 = .43133.

Cos 25° 33′ = .90221; cos 25° 34′ = .90208; difference = .00013; .00013 × $\frac{7}{60}$ = .00002, nearly. Hence, cos 25° 33′ 7″ = .90221 − .00002 = .90219.

108. Case II.—*When two sides are given:*

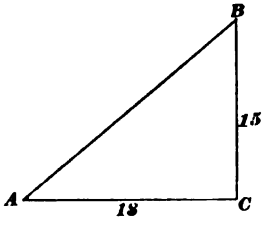

FIG. 70.

EXAMPLE 1.—In the right triangle $A\,C\,B$, Fig. 70, right-angled at C, $A\,C = 18$ and $B\,C = 15$; find $A\,B$ and the angles A and B.

SOLUTION.—As neither of the two acute angles is given, one of the angles must be found by making use of the definition of one of the functions of the angle. Considering the angle A, we have: side opposite equals 15 and the side adjacent equals 18; hence, we may use the definition of either the tangent or cotangent. Using the definition of the tangent,

$$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{15}{18} = .83333.$$

To find the angle whose tangent is .83333, we have: Tangent of next less angle is .83317 = tan 39° 48′; tangent of the next greater angle is .83366; difference is .00049. The difference between .83317, the tangent of the smaller angle, and .83333, the given tangent, is .83333 − .83317 = .00016. Hence, $\frac{16}{49}$ × 60 = 19.6″, and the angle whose tangent is .83333 = 39° 48′ 19.6″ = angle A.

Angle B = 90° − 39° 48′ 19.6″ = 50° 11′ 40.4″.

To find the hypotenuse $A\,B$, use rule **2** or **4**; using rule **2**,

$$\text{Hypotenuse} = \frac{\text{side opposite}}{\text{sine}} = \frac{15}{\sin 39° \, 48' \, 19.6''} = \frac{15}{.64018} = 23.43.$$

$$\text{Ans.}\begin{cases}\text{Angle } A = 39° \, 48' \, 19.6''.\\ \text{Angle } B = 50° \, 11' \, 40.4''.\\ A\,B = 23.43.\end{cases}$$

EXAMPLE 2.—In the right triangle $A\,C\,B$, Fig. 71 right-angled at C, $A\,C$ = .024967 mile and $A\,B$ = .04792 mile; find the other parts.

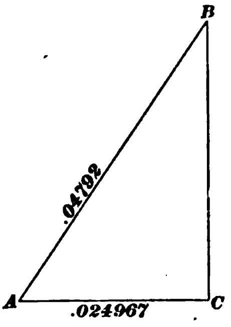

FIG. 71.

SOLUTION.—Here the hypotenuse and the side adjacent are given; hence, using the definition of the cosine,

$$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{.024967}{.04792} = .52101.$$

The angle whose cosine is .52101 = 58° 36′ = angle A. Angle $B = 90° - 58°\ 36' = 31°\ 24'$.

To find side $B\,C$, use rule **5.**

Side opposite A = side adjacent × tan A, or $B\,C = .024967 \times 1.63826 = .0409$ mile.

$$\text{Ans.} \begin{cases} \text{Angle } A = 58°\ 36'. \\ \text{Angle } B = 31°\ 24'. \\ B\,C = .0409 \text{ mi.} \end{cases}$$

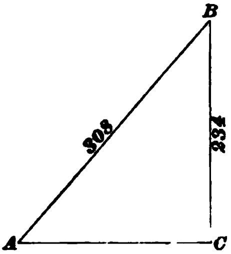

FIG. 72.

EXAMPLE 3.—In the right triangle $A\,C\,B$, Fig. 72, right-angled at C, $A\,B = 308$ feet and $B\,C = 234$ feet; find the other parts.

SOLUTION.—Here the hypotenuse and the side opposite are given; hence, using the definition of sine,

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{234}{308} = .75974.$$

The angle whose sine is .75974 = 49° 26′ 28″, nearly, = angle A. Angle $B = 90° - 49°\ 26'\ 28'' = 40°\ 33'\ 32''$.

To find $A\,C$, rule **1, 3, 5,** or **6** may be used. Using rule **6,**

side adjacent angle A = side opposite × cot A, or $A\,C = 234 \times .85586 = 200.27$ feet.

$$\text{Ans.} \begin{cases} \text{Angle } A = 49°\ 26'\ 28''. \\ \text{Angle } B = 40°\ 33'\ 32''. \\ A\,C = 200.27 \text{ ft.} \end{cases}$$

EXAMPLES FOR PRACTICE.

1. In the right triangle $A\,C\,B$, right-angled at C, the hypotenuse $A\,B = 40$ inches and angle $A = 28°\ 14'\ 14''$. Solve the triangle.

$$\text{Ans.} \begin{cases} \text{Angle } B = 61°\ 45'\ 46''. \\ A\,C = 35.24 \text{ in.} \\ B\,C = 18.92 \text{ in.} \end{cases}$$

2. In a right triangle $A\,C\,B$, right-angled at C, the side $B\,C$ = 10 feet 4 inches. If angle $A = 26°\ 59'\ 6''$, what do the other parts equal?

$$\text{Ans.} \begin{cases} \text{Angle } B = 63°\ 0'\ 54''. \\ A\,B = 22 \text{ ft. } 9\tfrac{1}{4} \text{ in., nearly.} \\ A\,C = 20 \text{ ft. } 3\tfrac{1}{4} \text{ in., nearly.} \end{cases}$$

3. In a right triangle $A\,C\,B$, the hypotenuse $A\,B = 60$ feet and the side $A\,C = 22$ feet. Solve the triangle.

$$\text{Ans.}\begin{cases}\text{Angle } A = 68^\circ\ 29'\ 22.2''.\\ \text{Angle } B = 21^\circ\ 30'\ 37.8''.\\ B\,C = 55.82 \text{ ft.}\end{cases}$$

4. In a right triangle $A\,C\,B$, right-angled at C, side $A\,C = .364$ foot and side $B\,C = .216$ foot. Solve the triangle.

$$\text{Ans.}\begin{cases}\text{Angle } A = 30^\circ\ 41'\ 7.5''.\\ \text{Angle } B = 59^\circ\ 18'\ 52.5''.\\ A\,B = .423 \text{ ft.}\end{cases}$$

OBLIQUE TRIANGLES.

109. When three parts of *any* triangle are given, one being a side, the remaining parts can be found by drawing a perpendicular from one angle to the opposite side, thus forming two right triangles. The parts of these right triangles can then be computed, and from them the parts of the required triangle can be found.

110. Caution.—*When dividing the triangle into two right triangles, care must be taken that the perpendicular be so drawn that one of the right triangles will have two known parts besides the right angle; otherwise the triangle cannot be solved.*

111. Case I.—*When the three known parts are a side and two angles, or two sides and the included angle:*

Example 1.—In Fig. 73, the angle $A = 46^\circ\ 14'$, the angle $B = 88^\circ\ 24'\ 11''$, and the side $A\,B = 21$ inches; find $A\,C$, $B\,C$, and the angle C.

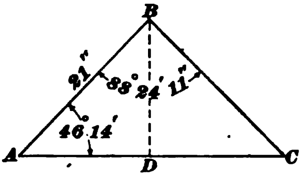

Fig. 73.

Solution.—Since the sum of all the angles of any triangle is 2 right angles, or 180° (Art. **48**), we can find the angle C by adding the two known angles and subtracting their sum from 180°.

$$88^\circ\ 24'\ 11'' + 46^\circ\ 14' = 134^\circ\ 38'\ 11''.$$

$$180^\circ - 134^\circ\ 38'\ 11'' = 45^\circ\ 21'\ 49'' = \text{angle } C.$$

From the vertex B, draw $B\,D$ perpendicular to $A\,C$. The triangle $A\,B\,C$ is now divided into two right triangles $A\,D\,B$ and $B\,D\,C$, both right-angled at D.

In the right triangle $A\,D\,B$, the angle A, the right angle D, and the hypotenuse $A\,B$ are known; find $B\,D$ and $A\,D$. Using rule **1**, side opposite, or $B\,D$, $= 21 \times \sin 46° 14' = 21 \times .72216 = 15.17$ inches, nearly.

Using rule **3**, side adjacent, or $A\,D$, $= 21 \times \cos 46° 14' = 21 \times .69172$, or $A\,D = 14.53$ inches, nearly.

In the right triangle $B\,D\,C$, the angle C and the side opposite, or $B\,D$, are known; find $B\,C$ and $D\,C$.

Using rule **2**, hypotenuse, or

$$B\,C = \frac{B\,D}{\sin 45° 21' 49''} = \frac{15.17}{.71158} = 21.32 \text{ inches, nearly.}$$

Using rule **3**, side adjacent, or $C\,D$, $= 21.32 \times \cos 45° 21' 49'' = 21.32 \times .70261 = 14.98$ inches.

Since $A\,D + D\,C = A\,C$, we have $14.53 + 14.98 = 29.51$ inches $= A\,C$.

$$\text{Ans.} \begin{cases} A\,C = 29.51 \text{ in.} \\ B\,C = 21.32 \text{ in.} \\ \text{Angle } C = 45° 21' 49''. \end{cases}$$

If, in the above example, the angle C had been given instead of the angle A, the dividing line should have been drawn from the angle A to the side $B\,C$, as in the following example:

Example 2.—In the triangle $A\,B\,C$, Fig. 74, given $A\,B = 18$ inches, angle $B = 60°$, and angle $C = 38° 42'$; find the other three parts.

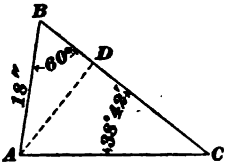

Fig. 74.

Solution.—In the triangle $A\,B\,C$, we have angle $A = 180° - (60° + 38° 42') = 81° 18'$.

From the vertex A, draw the line $A\,D$ perpendicular to $B\,C$, thus forming the right triangles $A\,D\,B$ and $A\,D\,C$.

In the triangle $A\,D\,B$, two parts (the side $A\,B$ and angle B) are known besides the right angle. To find $B\,D$, use rule **3**. $B\,D = 18 \times \cos 60° = 18 \times .5 = 9$ inches. To find $A\,D$, use rule **1**. $A\,D = 18 \times \sin 60° = 18 \times .86603 = 15.59$ inches.

In the right triangle $A\,D\,C$, $A\,D$ and the angle C are known.

To find $A\,C$, use rule **2**.

$$A\,C = \frac{A\,D}{\sin C} = \frac{15.59}{.62524} = 24.93 \text{ inches.}$$

To obtain $D\,C$, use rule **3**.

$$D\,G = A\,C \times \cos C = 24.93 \times .78043 = 19.46 \text{ inches.}$$

Since $B\,C = B\,D + D\,C$, $B\,C = 9 + 19.46 = 28.46$ inches.

$$\text{Ans.} \begin{cases} A\,C = 24.93 \text{ in.} \\ B\,C = 28.46 \text{ in.} \\ \text{Angle } A = 81° 18'. \end{cases}$$

Example 3.—In Fig. 75, $A B = 19$ inches, $A C = 23$ inches, and the included angle $A = 36° 3' 29''$; find the other two angles and the side $B C$.

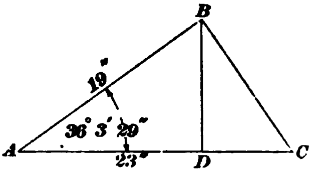

FIG. 75.

Solution.—From the vertex B, draw $B D$ perpendicular to $A C$, forming the two right triangles $A D B$ and $B D C$. In the right triangle $A D B$, $A B$ is known and also the angle A. Hence, by rule **1**,

$B D = 19 \times \sin 36° 3' 29'' = 19 \times .58861 = 11.18$ inches, nearly.

By rule **3**, $A D = 19 \times \cos 36° 3' 29'' = 19 \times .80842 = 15.36$ inches.

$A C - A D = 23 - 15.36 = 7.64$ inches $= D C$.

In the right triangle $B D C$, the two sides $B D$ and $D C$, about the right angle, are known; hence, from the definition of tangent,

$$\tan C = \frac{B D}{D C} = \frac{11.18}{7.64} = 1.46335, \text{ and angle } C = 55° 39' 10''.$$

Applying rule **2**,

$$B C = \frac{B D}{\sin 55° 39' 10''} = \frac{11.18}{.82564} = 13.54 \text{ inches.}$$

Angle $B = 180° - (36° 3' 29'' + 55° 39' 10'') = 180° - 91° 42' 39'' = 88° 17' 21''$.

Ans. $\begin{cases} \text{Angle } C = 55° 39' 10''. \\ \text{Angle } B = 88° 17' 21''. \\ \text{Side } B C = 13.54 \text{ in.} \end{cases}$

112. Case II.—*When the three known parts are two sides and an angle opposite one of them:*

For this case there are, in general, two solutions. This is readily seen by referring to Fig. 76. Suppose the given parts are the sides $A B$ and $B C$ and the angle A opposite the side $B C$. We construct the triangle by first drawing the lines $A E$ and $A F$ in such a manner that the angle A shall be of the required size, and then lay off the distance $A B$ along $A E$ to represent the length of the side $A B$. To draw the side $B C$, we take the point B as a center, and with a radius equal to the length of $B C$, we describe the arc $C C'$ and draw $B C$ and $B C'$. The required triangle may be either $A B C$ or

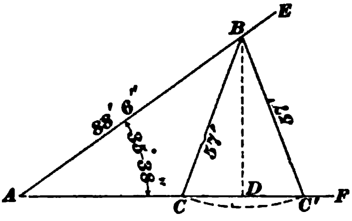

FIG. 76.

$A B C'$. In practice, the conditions will indicate to us which triangle to select; but when the two sides and the angle opposite one of them only are given and no other condition is stated, it is necessary to solve both triangles, which is readily done as follows:

First solve the triangle $A B C$. To do this, find the length of the perpendicular $B D$ by applying rule **1** to angle A ($B D = A B \times \sin A$); find angle $B C D$ by applying definition of sine to angle $B C D$ $\left(\sin B C D = \frac{B D}{C B}\right)$; find $C D$ by applying rule **3** ($C D = C B \times \cos B C D$); find $A D$ by applying rule **3** ($A D = A B \times \cos A$). We now know all that is necessary to determine the unknown parts of both triangles. For, the angle $A C B$ is the supplement (see Art. **27**) of the angle $B C D$, and is therefore equal to $180° -$ angle $B C D$; the angle $A B C = 180° -$ (angle $B A C +$ angle $A C B$); the side $A C = A D - C D$; since $C B C'$ is an isosceles triangle, angle $B C D = B C' D$ and $C' D = C D$; $A C' = A D + C' D$; and, finally, angle $A B C' = 180° -$ (angle $A +$ angle C').

113. While, in general, there are two solutions to examples falling under Case II, there may be no solution or only one solution, depending on the length of the side $B C$.

a. If $B C$ is less than the perpendicular $B D$, the arc $C C'$ will not touch the side $A F$ of the angle, and no triangle can be formed; hence, in this instance there is no solution.

b. If $B C$ is just equal to $B D$, the arc $C C'$ will touch $A F$ at only one point; only one triangle can be formed—a right triangle—and there is one solution.

c. If $B C$ is greater than $B D$ and less than $A B$, the arc $C C'$ will cut $A F$ between A and D, and also to the right of D; this gives two triangles and two solutions.

d. If $B C$ is just equal to $A B$, the arc $C C'$ will cut $A F$ at A and at a point at a distance $A D$ to the right of D; this gives one triangle and one solution.

e. If $B C$ is greater than $A B$, the arc $C C'$ will not cut $A F$ between A and D, but will cut $A F$ at a point to the right of D; hence, but one triangle can be formed and there is but one solution.

EXAMPLE.—In Fig. 76, $A B = 88$ feet 6 inches, $B C = 57$ feet, and angle $A = 35° \, 0' \, 38''$; find the other parts.

SOLUTION.—Applying the various steps in the order given in Art. **112,** we have by rule **2,** $B D = 88$ feet 6 inches $\times \sin 35° \, 0' \, 38'' = 88.5 \times .57373 = 50.78$ feet.

$\sin B C D = \frac{B D}{B C} = \frac{50.78}{57} = .89088$; whence, angle $B C D = 62° \, 59' \, 4.3''$.

By rule **3,** $C D = 57 \times \cos 62° \, 59' \, 4.3'' = 57 \times .45423 = 25.89$ feet.

By rule **3,** $A D = 88.5 \times \cos 35° \, 0' \, 38'' = 88.5 \times .81905 = 72.49$ feet.

We now have the data necessary for obtaining the required parts of the triangle $A B C$. Since the angle $B C D = 62° \, 59' \, 4.3''$, the adjacent angle $A C B = 180° - 62° \, 59' \, 4.3'' = 117° \, 0' \, 55.7''$. Also, angle $A B C = 180° - (35° \, 0' \, 38'' + 117° \, 0' \, 55.7'') = 180° - 152° \, 1' \, 33.7'' = 27° \, 58' \, 26.3''$. Since $A D = 72.49$ feet and $C D = 25.89$ feet, $A C = 72.49 - 25.89 = 46.6$ feet.

For the triangle $A B C'$, angle $C' = 62° \, 59' \, 4.3''$ and angle $A B C' = 180° - (35° \, 0' \, 38'' + 62° \, 59' \, 4.3'') = 82° \, 0' \, 17.7''$. $A C' = 72.49 + 25.89 = 98.38$ feet.

$$\text{Ans.} \begin{cases} \text{Angle } C = 117° \, 0' \, 55.7''. \\ \text{Angle } B = 27° \, 58' \, 26.3''. \\ \text{Side } A C = 46.6 \text{ ft.} \\ \text{Angle } A B C' = 82° \, 0' \, 17.7''. \\ \text{Angle } C' = 62° \, 59' \, 4.3''. \\ \text{Side } A C' = 98.38 \text{ ft.} \end{cases}$$

114. Case III.—*When the three sides are given, to find the angles:*

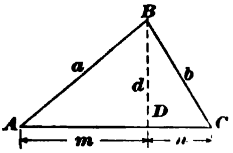

FIG. 77.

This case is solved by drawing a line from the vertex of the angle opposite the longest side, perpendicular to that side, as $B D$ in Fig. 77. The parts m and n of the side $A C$ are then determined from the following proportion:

$$m + n \text{ (or } A C) : a + b = a - b : m - n.$$

This gives the value of $m - n$. The value of $m + n = A C$ is already known, and from the two, m and n may be determined by the principles of arithmetic, as explained below. Having m and n, therefore, the right triangles $A B D$ and $C B D$ may be solved.

Having found the value of $m - n$ and knowing the value of $m + n$, the values of m and n may be determined as follows: It is a principle of arithmetic that if the sum of two numbers and their difference be given, *the greater of the two numbers is equal to one-half the sum of their sum and their difference, and the less of two numbers is equal to one-half the difference between their sum and their difference.* For example, suppose that the sum of two numbers is 22 and their difference is 8. Then, the greater number is $(22 + 8) \div 2 = 15$, and the less number is $(22 - 8) \div 2 = 7$. Therefore, letting m be the greater number and n the less number, $m + n$ represents their sum and $m - n$ their difference; whence,

$$m = \frac{(m + n) + (m - n)}{2},$$

$$n = \frac{(m + n) - (m - n)}{2}.$$

Example.—Given, a triangle whose sides are 17 feet 3 inches, 21 feet, and 32 feet long. Find the angles.

Solution.— $m + n$, the longest side, $= 32$ feet.

$a + b$, the sum of the two shorter sides, $= 21 + 17.25 = 38.25$ feet.

$a - b$, the difference of the two shorter sides, $= 3.75$ feet. Hence,

$$32 : 38.25 = 3.75 : m + n, \text{ or } m - n = \frac{38.25 \times 3.75}{32} = 4.48 \text{ feet.}$$

Then, $$m = \frac{(m + n) + (m - n)}{2} = \frac{32 + 4.48}{2} = 18.24 \text{ feet;}$$

and $$n = \frac{(m + n) - (m - n)}{2} = \frac{32 - 4.48}{2} = 13.76 \text{ feet.}$$

Now, referring to the last figure, we have, in the triangle $A\,D\,B$ side $a = 21$ feet and $m = 18.24$ feet; whence, by definition of cosine,

$$\cos A = \frac{18.24}{21} = .86857, \text{ or } A = 29^\circ\, 42'\, 25.7''.$$

In triangle $C\,B\,D$, side $b = 17.25$ feet and $n = 13.76$ feet; whence,

$$\cos C = \frac{13.76}{17.25} = .79768, \text{ or } C = 37^\circ\, 5'\, 26.7''.$$

Angle $A\,B\,C = 180^\circ - (29^\circ\, 42'\, 25.7'' + 37^\circ\, 5'\, 26.7'') = 113^\circ\, 12'\, 7.6''$.

$$\text{Ans.} \left\{ \begin{array}{l} \text{Angle } A = 29^\circ\, 42'\, 25.7''. \\ \text{Angle } B = 113^\circ\, 12'\, 7.6''. \\ \text{Angle } C = 37^\circ\, 5'\, 26.7''. \end{array} \right.$$

EXAMPLES FOR PRACTICE.

1. Given, an oblique triangle ABC, in which side $AB = 21$ feet, angle $A = 22^\circ\ 10'\ 16''$, and angle $B = 78^\circ\ 24'\ 24''$. Find the other parts.

$$\text{Ans.}\begin{cases}\text{Angle } C = 79^\circ\ 25'\ 20''.\\ AC = 20.93 \text{ ft.}\\ BC = 8.06 \text{ ft.}\end{cases}$$

2. Given, a triangle ABC, in which $AB = 32$ inches, angle $B = 54^\circ\ 16'$, and angle $C = 58^\circ\ 18'\ 9''$. Find the other parts.

$$\text{Ans.}\begin{cases}\text{Angle } A = 67^\circ\ 25'\ 51''.\\ AC = 30.53 \text{ in.}\\ BC = 34.73 \text{ in.}\end{cases}$$

3. In a triangle ABC, $AB = 20$ feet 6 inches, $BC = 16$ feet, and angle $B = 46^\circ\ 40'\ 42''$. Find the values of the other parts.

$$\text{Ans.}\begin{cases}\text{Angle } A = 50^\circ\ 42'\ 51''.\\ \text{Angle } C = 82^\circ\ 36'\ 27''.\\ AC = 15.04 \text{ ft.}\end{cases}$$

4. In a triangle ABC, $AC = 100$ feet, $BC = 60$ feet, and angle $A = 20^\circ$. Solve the triangle.

$$\text{Ans.}\begin{cases}\text{Angle } B = 34^\circ\ 45'\ 7.5'', \text{ or}\\ \quad \text{angle } B = 145^\circ\ 14'\ 52.5''.\\ \text{Angle } C = 125^\circ\ 14'\ 52.5''.\\ AB = 143.268 \text{ ft., or } AB\\ \quad = 44.67 \text{ ft.}\end{cases}$$

5. In a triangle ABC, $AB = 98$ inches, $BC = 140$ inches, and $AC = 210$ inches. Compute the angles A, B, and C.

$$\text{Ans.}\begin{cases}A = 34^\circ\ 2'\ 52.5''.\\ B = 122^\circ\ 52'\ 40.2''.\\ C = 23^\circ\ 4'\ 27.3''.\end{cases}$$

MENSURATION.

115. **Mensuration** is that part of geometry which treats of the measurement of lines, surfaces, and solids.

MENSURATION OF PLANE SURFACES.

116. The **area** of a surface is expressed by the number of unit squares it will contain.

117. A **unit square** is the square whose side is equal in length to the unit. For example, if the unit is 1 inch, the unit square is the square whose sides measure 1 inch in length, and the area would be expressed by the number of square inches that the surface contains. If the unit were

1 foot, the unit square would measure 1 foot on each side, and the area would be the number of square feet that the surface contains, etc. The square that measures 1 inch on a side is called a **square inch,** and the one that measures 1 foot on a side is called a **square foot.** Square inch and square foot are abbreviated to sq. in. and sq. ft., or are indicated by □″ and □′.

THE TRIANGLE.

118. Rule.—*The area of any triangle equals one-half the product of the base and the altitude.*

Letting b be the base, h the altitude, and A the area,

$$A = \frac{b\,h}{2}.$$

If the triangle is a right triangle, one of the short sides may be taken as the base, and the other short side as the altitude; hence, *the area of a right triangle is equal to one-half the product of the two short sides.*

Example.—What is the area of a triangle whose base is 18 feet and altitude 7 feet 9 inches?

Solution.— 9 inches = $\frac{9}{12}$ foot = $\frac{3}{4}$ foot; hence,

$$A = \frac{b\,h}{2} = \frac{18 \times 7\frac{3}{4}}{2} = 69\tfrac{3}{4} \text{ sq. ft.} \quad \text{Ans.}$$

119. The area of any triangle may be found, when the length of each side is known, by means of the following formula, in which a, b, and c represent the lengths of the sides, s half the sum of the lengths, and A the area of the triangle:

$$A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}, \text{ where } s = \frac{a+b+c}{2}.$$

Example.—What is the area of a triangle having two sides 19.8 feet long and one side 28 feet long?

Solution.—It is immaterial which side is called a, b, or c. Applying the formula, $s = \frac{a+b+c}{2} = \frac{28 + 19.8 + 19.8}{2} = 33.8$, the half sum; taking b and c as the short sides, $s - a = 33.8 - 28 = 5.8$ and $s - b$ and $s - c$ are each $33.8 - 19.8 = 14$. Then,

$$A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)} = \sqrt{33.8 \times 5.8 \times 14 \times 14}$$
$$= 196+ \text{ sq. ft.} \quad \text{Ans.}$$

THE QUADRILATERAL.

120. A **parallelogram** is a quadrilateral whose opposite sides are parallel. There are four kinds of parallelograms: the *square*, the *rectangle*, the *rhombus*, and the *rhomboid*.

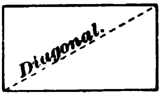

FIG. 78.

121. A **rectangle,** Fig. 78, is a parallelogram whose angles are all right angles.

122. A **square,** Fig. 79, is a rectangle, all of whose sides are equal.

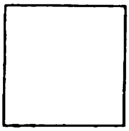

FIG. 79.

FIG. 80.

123. A **rhomboid,** Fig. 80, is a parallelogram whose opposite sides only are equal and whose angles are not right angles.

124. A **rhombus,** Fig. 81, is a parallelogram having equal sides and whose angles are not right angles.

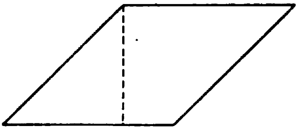

FIG. 81.

FIG. 82.

125. A **trapezoid,** Fig. 82, is a quadrilateral which has only two of its sides parallel.

126. A **trapezium,** Fig. 83, is a quadrilateral having no two sides parallel.

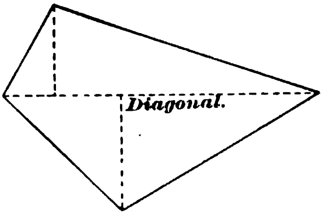

FIG. 83.

127. The **altitude** of a parallelogram, or of a trapezoid, is the perpendicular distance between the parallel sides. See dotted line in Figs. 80, 81, and 82.

128. A **diagonal** is a straight line drawn from the vertex of any angle of a quadrilateral to the vertex of the angle opposite; a diagonal divides a quadrilateral into two triangles. See Figs. 78 and 83.

A diagonal divides a parallelogram into two *equal* and *similar* triangles.

129. To find the area of a parallelogram :

Rule.—*The area of any parallelogram equals the product of the base and the altitude.*

Let b be the base, h the altitude, and A the area; then,

$$A = b\,h.$$

Example.—What is the area of a parallelogram whose base is 12 feet and altitude $7\frac{1}{2}$ feet ?

Solution.—Applying the formula, $A = bh = 12 \times 7\frac{1}{2} = 90$ sq. ft. Ans.

If the area and one dimension are given, the other may be found by dividing the area by the known dimension. If the parallelogram is a square, and its area is given, the length of a side is found by extracting the square root of the area ; that is, $b = \sqrt{A}$.

130. To find the area of a trapezoid :

Rule.—*The area of a trapezoid equals one-half the sum of the parallel sides multiplied by the altitude.*

Let a and b represent the lengths of the parallel sides and h the altitude; then,

$$A = \left(\frac{a+b}{2}\right)h.$$

Example.—What is the area of a trapezoid whose parallel sides are 9 feet and 15 feet and whose altitude is 6 feet 7 inches ?

Solution.— 6 feet 7 inches $= 6\frac{7}{12}$ feet. Using the formula,

$$A = \left(\frac{a+b}{2}\right)h = \frac{9+15}{2} \times 6\tfrac{7}{12} = 79 \text{ sq. ft.} \quad \text{Ans.}$$

THE CIRCLE.

131. To find the circumference, diameter, or radius of a circle:

Rule.—*The circumference of a circle equals the diameter multiplied by 3.1416.*

Rule.—*The diameter of a circle equals the circumference divided by 3.1416; the radius equals the circumference divided by 2 × 3.1416.*

Let d be the diameter, r the radius, and c the circumference,

$$c = \pi d = 2 \pi r;$$

and

$$d = \frac{c}{\pi}; \text{ or } r = \frac{c}{2 \pi}.$$

EXAMPLE 1.—What is the circumference of a circle whose diameter is 15 inches?

SOLUTION.—Using the formula, $c = \pi d = 3.1416 \times 15 = 47.12$ in. Ans.

EXAMPLE 2.—What is the diameter of a circle whose circumference is 65.973 inches?

SOLUTION.—Using the formula, $d = \frac{c}{\pi} = \frac{65.973}{3.1416} = 21$ in. Ans.

The number 3.1416 is the ratio of the circumference of a circle to its diameter; it is represented very frequently by the Greek letter π, pronounced "pi." Its value has been calculated to over 700 decimal places, but the value here given is the one most generally used, four decimal places being sufficient for all practical purposes. The values $\frac{1}{4}\pi$, or .7854, and $\frac{1}{6}\pi$, or .5236, are frequently used farther on.

132. To find the length of an arc of a circle:

Rule.—*The length of an arc of a circle equals the circumference of the circle of which the arc is a part multiplied by the number of degrees in the arc and the product divided by 360.*

Let l be the length of arc, c the circumference, d the diameter of the circle, and n the number of degrees in the arc; then,

$$l = \frac{\pi d n}{360}.$$

Example.—What is the length of an arc of 24°, the radius of the circle being 18 inches?

Solution.— $18 \times 2 = 36$ inches, the diameter of the circle. Using the formula, $l = \frac{\pi d n}{360} = \frac{3.1416 \times 36 \times 24}{360} = 7.54$ in., length of arc. Ans.

133. When only the chord of the arc and the height of segment (that is, *A B* and *C D*, Fig. 84) are given, the following closely approximate formula may be used:

Let c be the length of chord, h the height of segment, and l the length of arc; then,

$$l = \frac{4\sqrt{c^2 + 4h^2} - c}{3}.$$

Example.—If *A B*, Fig. 84, is 5 feet and *C D* is 1 foot, what is the length of arc *A D B*?

Solution.—Applying the formula,

$$l = \frac{4\sqrt{c^2 + 4h^2} - c}{3} = \frac{4\sqrt{25 + 4} - 5}{3} = 5.51 \text{ ft. Ans.}$$

When the quotient obtained by dividing the chord by the height is less than 4.8, that is, when $\frac{c}{h}$ is less than 4.8, the formula does not work well, the results not being sufficiently exact. In such a case, bisect the arc and then apply the formula.

134. To find the area of a circle:

Rule.—*Square the diameter and multiply by .7854; or, square the radius and multiply by 3.1416.*

Let A be the area; then,

$$A = \frac{1}{4}\pi d^2 = .7854\, d^2; \text{ or, } A = \pi r^2 = 3.1416\, r^2.$$

Example.—What is the area of a circle whose diameter is 15 inches?

Solution.— $15^2 = 225$. Using the formula, $A = .7854\, d^2 = .7854 \times 225 = 176.72$ sq. in. Ans.

135. Given the area of a circle, to find its diameter:

Rule.—*Divide the area by .7854 and extract the square root of the quotient.*

Expressed as a formula, the rule is

$$d = \sqrt{\frac{A}{.7854}} = 1.1284\sqrt{A} = \sqrt{\frac{4A}{\pi}}.$$

EXAMPLE.—The area of a circle = 17,671.5 square inches. What is its diameter in feet ?

SOLUTION.—Using the formula,

$$d = \sqrt{\frac{A}{.7854}} = \sqrt{\frac{17{,}671.5}{.7854}} = 1.1284\sqrt{17{,}671.5} = 150 \text{ inches.}$$

$$\frac{150}{12} = 12\tfrac{1}{2} \text{ ft. diameter. Ans.}$$

136. To find the area of a flat circular ring:

Rule.—*Subtract the area of the smaller circle from that of the larger; the difference is the area of the ring.*

Let d equal the longer diameter, d_1 the shorter diameter, and A the area of ring; then,

$$A = .7854\,d^2 - .7854\,d_1^2 = .7854\,(d^2 - d_1^2).$$

EXAMPLE.—What is the area of a ring whose longer and shorter diameters are 6.5 feet and 4 feet, respectively ?

SOLUTION.—Applying the formula,

$$A = .7854\,(6.5^2 - 4^2) = .7854 \times 26.25 = 20.62 \text{ sq. ft. Ans.}$$

If one diameter and the area of the ring are known, the other diameter may be found by adding to or subtracting from the area of the given circle that of the ring, and finding the diameter corresponding to the resulting area.

137. To find the area of a sector:

Rule.—*Divide the number of degrees in the arc of the sector by 360. Multiply the result by the area of the circle of which the sector is a part.*

Let n be the number of degrees in the arc, A the area of circle, d the diameter of circle, and A' the area of sector; then,

$$A' = \frac{nA}{360} = .0021817\,d^2 n.$$

Example.—The number of degrees in the angle formed by drawing radii from the center of a circle to the extremities of the arc of the circle is 75. The diameter of the circle is 12 inches; what is the area of the sector?

Solution.— $12^2 \times .7854 = 113.1$ square inches, nearly, area of circle. Applying the formula,

$$A' = \frac{n\,A}{360} = \frac{75 \times 113.1}{360} = 23.56 \text{ sq. in.} \quad \text{Ans.}$$

138. If the length of arc and radius of a sector are given, the following rule may be used:

Rule.—*The area of a sector is equal to one-half the product of the radius and length of arc.*

Let l be the length of arc, r the radius, and A' the area; then,

$$A' = \frac{l\,r}{2}.$$

Example.—If the radius of an arc is 5 feet and the length of arc is 4 feet, what is the area of the sector?

Solution.—Applying the formula,

$$A' = \frac{l\,r}{2} = \frac{4 \times 5}{2} = 10 \text{ sq. ft.} \quad \text{Ans.}$$

139. To find the area of a segment of a circle:

Rule.—*Draw radii from the center of the circle to the extremities of the arc of the segment; find the area of the sector thus formed, subtract from this the area of the triangle formed by the radii and the chord of the arc of the segment; the result is the area of the segment.*

In problems requiring the area of the segment, the chord $A\,B$, Fig. 84, may be given, or the height of the segment $C\,D$, or the angle V; if any one of these three is given and the radius of the circle is known, the area can be found.

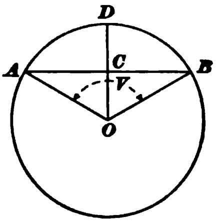

Fig. 84.

Also, if any two are given, the radius can be found.

EXAMPLE 1.—If the diameter of the circle is 10 inches and the chord of the segment is 7 inches, what is the area of the segment?

SOLUTION.—In the above figure, suppose that the chord $A\,B = 7$ inches and the diameter $= 10$ inches; draw $O\,A$, $O\,B$, and a radius perpendicular to the chord, thus dividing $A\,B$ into two equal parts (see Art. **70**). The triangle $A\,O\,B$ is now divided into two equal right triangles $A\,C\,O$ and $B\,C\,O$, in which the hypotenuse = radius $= \frac{10}{2} = 5$ inches, and one side $A\,C = B\,C = \frac{7}{2}$, or $3\frac{1}{2}$ inches.

$$\text{Sin } C\,O\,B = \frac{C\,B}{O\,B} = \frac{3\frac{1}{2}}{5} = .70000, \text{ and angle } C\,O\,B = 44°\ 26', \text{ nearly.}$$

Angle $A\,O\,B = 44°\ 26' \times 2 = 88°\ 52'$. $C\,O = O\,B \times \cos C\,O\,B = 5 \times .71407 = 3.57$ inches.

$$\text{Area of sector} = 10^2 \times .7854 \times \frac{88\frac{52}{60}}{360} = 19.39 \text{ sq. in., nearly.}$$

$$\text{Area of triangle} = \frac{7 \times 3.57}{2} = 12.5 \text{ sq. in., nearly.}$$

$19.39 - 12.5 = 6.89$ sq. in., the area of segment. Ans.

EXAMPLE 2.—Given the chord of the arc of a segment = 7 inches and the height of the segment = 1.43 inches, to find the radius.

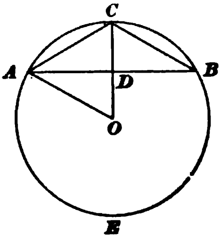

FIG. 85.

SOLUTION.—Suppose that in Fig. 85, $A\,C\,B\,E$ is a circle struck with the required radius, that the chord $A\,B = 7$ inches, and that the height $C\,D$ of the segment $= 1.43$ inches. Join C with A and B, and the right triangle $A\,D\,C = B\,D\,C$.

$$\text{Tan } C\,B\,D = \frac{C\,D}{B\,D} = \frac{1.43}{3.5} = .40857.$$

$$\text{Angle } C\,B\,D = 22°\ 13\tfrac{1}{2}', \text{ nearly.}$$

Since $C\,B\,D$ or its equal $C\,B\,A$ is an inscribed angle (see Art. **66**), it is measured by one-half the intercepted arc $A\,C$; hence, the number of degrees in arc $A\,C = 22°\ 18\frac{1}{2}' \times 2 = 44°\ 27'$, or the number of degrees in the angle $A\,O\,C$.

In the right triangle $A\,D\,O$,

$$AO = \frac{\text{side opposite}}{\sin A\,O\,D} = \frac{A\,D}{\sin A\,O\,C} = \frac{3.5}{.70029} = 5 \text{ in., nearly. Ans.}$$

NOTE.—The principles explained in the two preceding examples may be used in solving problems relating to length of radius, chord, sub-chord (chord, as $A\,C$, of half the arc $A\,B$), height of segment, etc. These all involve the principle of the right triangle.

REGULAR POLYGONS.

140. A **regular polygon** may be divided into as many equal isosceles triangles as there are sides, by drawing lines from the center to the angles. Each of the angles formed at the center is equal to 360° divided by the number of sides.

To find the area of a regular polygon:

Rule.—*Multiply together the number of sides, the square of the length of a side, and the cotangent of* **one-half** *the central angle, and divide the product by 4. The result will be the area of the regular polygon.*

Let A be the area, n the number of sides, l the length of a side, and x *one-half* the central angle included between two lines drawn from the center to the extremities of a side; then,

$$A = \frac{n\,l^2 \cot x}{4}.$$

Example.—What is the area of a regular decagon having sides 5 feet long?

Solution.—Here n is 10; l is 5 feet; x is $\frac{360}{10} + 2 = 18°$; cot 18° = 3.07768; whence,

$$A = \frac{n\,l^2 \cot x}{4} = \frac{10 \times 5 \times 5 \times 3.07768}{4} = 192.35 \text{ sq. ft.} \quad \text{Ans.}$$

141. The area of a regular polygon whose sides are known may also be found in the following manner:

Rule.—*Square the length of a side and multiply by the proper multiplier in the subjoined table.*

Name.	No. Sides.	Multiplier.	Name.	No. Sides.	Multiplier.
Equilateral Triangle	3	0.4330	Octagon	8	4.8284
Square	4	1.0000	Nonagon	9	6.1818
Pentagon	5	1.7205	Decagon	10	7.6942
Hexagon	6	2.5981	Undecagon	11	9.3656
Heptagon	7	3.6339	Dodecagon	12	11.1960

EXAMPLE.—What is the area of a regular octagon having sides 8 feet long?

SOLUTION.— $8^2 = 64$; multiplying 64 by the corresponding tabular number, 4.8284, the area is found to be $64 \times 4.8284 = 309.02$ sq. ft. Ans.

THE ELLIPSE.

142. An **ellipse** is a plane figure bounded by a curved line, to any point of which the sum of the distances from two fixed points within, called the **foci,** is equal to the sum of the distances from the foci to any other point on the curve.

In Fig. 86, let A and B be the foci and let C and D be any two points on the perimeter. Then, according to the above definition, $AC + CB = AD + DB$, and both these sums are also equal to the long diameter FE.

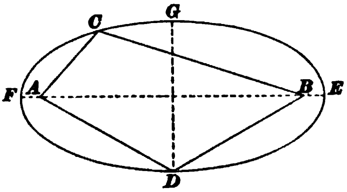

FIG. 86.

The foci may be located from G or D as a center by striking arcs cutting FE at A and B, using a radius equal to one-half of FE.

The long diameter of an ellipse, as FE, Fig. 86, is called the **major axis;** the short diameter, as GD, is called the **minor axis.**

143. To find the periphery (perimeter) of an ellipse: There is no exact method, but the following formula gives values very nearly exact. In the formula,

$\pi = 3.1416$;
$C =$ periphery;
$a =$ half the major axis;
$b =$ half the minor axis;

$$D = \frac{a - b}{a + b}.$$

$$C = \pi\,(a + b)\,\frac{64 - 3\,D^4}{64 - 16\,D^2}.$$

EXAMPLE.—What is the periphery of an ellipse whose axes are 10 inches and 4 inches?

Solution.—Applying the formula, $a = 5$, $b = 2$, $D = \frac{5-2}{5+2} = \frac{3}{7}$.

Then, $C = 3.1416\,(5+2)\,\frac{64 - 3(\frac{3}{7})^4}{64 - 16(\frac{3}{7})^2} = 23.014$ in. Ans.

144. To find the area of an ellipse:

Rule.—*The area of an ellipse is equal to the product of its two diameters multiplied by .7854.*

Let A be the longer diameter, or major axis; B the shorter diameter, or minor axis; and S the area; then,

$$S = \tfrac{1}{4}\pi A\,B = .7854\,A\,B.$$

Example.—What is the area of an ellipse whose diameters are 10 inches and 6 inches?

Solution.—Applying the formula, $S = .7854\,A\,B = .7854 \times 10 \times 6 = 47.12$ sq. in area. Ans.

EXAMPLES FOR PRACTICE.

1. What is the area in square feet of a rhombus whose base is 84 inches and whose altitude is 3 feet? Ans. 21 sq. ft.

2. One side of a room is 16 feet long. If the floor contains 240 square feet, what is the length of the other side? Ans. 15 ft.

3. How many square feet in a board 12 feet long, 18 inches wide at one end and 12 inches wide at the other end? Ans. 15 sq. ft.

4. How many square yards of plastering will be required for the ceiling and walls of a room 10 ft. × 15 ft. and 9 feet high? The room contains one door 3½ ft. × 7 ft., three windows 3½ × 6 ft., and a baseboard 8 inches high. Ans. 53.5 sq. yd.

5. What is the area of a triangle whose base is 10 feet 6 inches long and whose altitude is 18 feet? Ans. 94.5 sq. ft.

6. The area of a triangle is 16 square inches. If the altitude is 4 inches, what does the base measure? Ans. 8 in.

7. The upper side of a trapezium is 16 inches long and the lower side 14 inches. If the figure be divided into two triangles by a diagonal whose altitudes, drawn from their vertexes to the two given sides as bases, are 17 inches and 3 inches, respectively, what is the area of the trapezium? Ans. 157 sq. in.

8. Find the area of a circle 2 feet 3 inches in diameter. Ans. 3.976 sq. ft.

9. A carriage wheel was observed to make 71⅝ turns while going 300 yards. What was its diameter? Ans. 4 ft., nearly.

10. Required, the diameter of a circle whose area is 2,004 square inches. Ans. 50.51 in.

11. Required, the area of a regular pentagon inscribed in a circle whose diameter is 20 inches. Ans. 237.77 sq. in.

12. The number of degrees in the angle formed by drawing radii from the center of a circle to the extremities of the arc of the circle is 84. The diameter of the circle is 17 inches; what is the area of the sector? Ans. 52.96 sq. in.

13. Given, the chord of the arc of a segment = 24 inches and the height of the segment = 6.5 inches, to find (*a*) the diameter of the circle, and (*b*) the area of the segment.

Ans. { (*a*) 28.654 in.
(*b*) 109.87 sq. in.

14. (*a*) What is the perimeter of an ellipse whose axes are 15 inches and 9 inches, and (*b*) what is the area?

Ans. { (*a*) 38.29 in.
(*b*) 106.03 sq. in.

145. To find the area of any plane figure bounded by straight or curved lines:

Rule.—*The area of any plane figure may be found by dividing it into triangles, quadrilaterals, circles or parts of circles, and ellipses, finding the area of each part separately and adding them together.*

EXAMPLE 1.—The diagonal of a trapezium is 15 feet. The altitudes drawn from the vertexes of the two triangles to this diagonal as a base are 6 feet 8 inches and 4 feet 9 inches, respectively. What is the area of the trapezium?

SOLUTION.— 8 inches = $\frac{8}{12}$ foot = $\frac{2}{3}$ foot. $\frac{15 \times 6\frac{2}{3}}{2} = 50$ square feet = the area of one triangle.

9 inches = $\frac{9}{12}$ foot = $\frac{3}{4}$ foot. $\frac{15 \times 4\frac{3}{4}}{2} = 35.63$ square feet = the area of the other triangle.

The area of the trapezium equals 50 + 35.63 = 85.63 sq. ft. Ans.

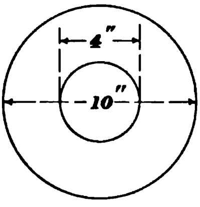

FIG. 87.

EXAMPLE 2.—What is the area of a flat circular ring, Fig. 87, whose outside diameter equals 10 inches and whose inside diameter equals 4 inches?

SOLUTION.—The area of the large circle = $10^2 \times .7854 = 78.54$ square inches; the area of the small circle = $4^2 \times .7854 = 12.57$ square inches.

$78.54 - 12.57 = 65.97$ sq. in., or the area. Ans.

Example 3.—What is the exact area in square inches of Fig. 88?

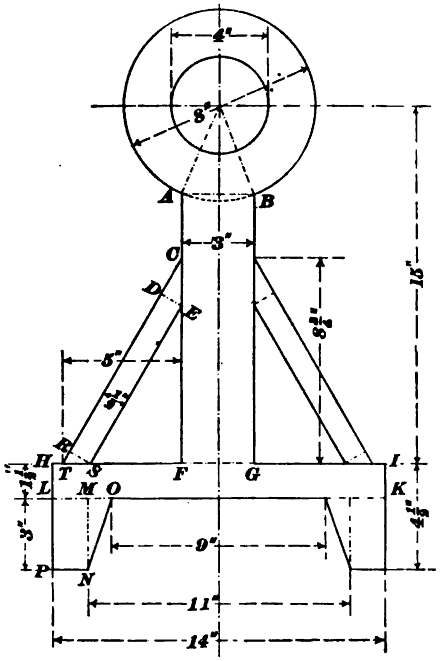

FIG. 88.

SOLUTION.—Divide the figure into rectangles, triangles, and parts of a circle, as shown by the dotted lines, then the total area equals 8-inch circle − 4-inch circle − segment AB + rectangle $ABGF$ + 2 times the triangle CDE + 2 times the triangle RST + 2 times the rectangle $DESR$ + the rectangle $HIKL$ + 2 times the rectangle $LMNP$ + 2 times the triangle MON.

$8^2 \times .7854 = 50.27$ □".
$4^2 \times .7854 = 12.57$ □".

The chord $AB = 3$ inches and the radius of the circle = 4 inches; hence, the sine of one-half the angle at center $= \frac{1.5}{4} = .375$, and one-half the angle at center $= 22° 1' 27''$, or angle at center $= 44° 2' 54'' = 44.05°$. Area of sector $= 50.27 \times \frac{44.05}{360} = 6.15$ □".

The altitude of the triangle $= 4 \times \cos 22° 1' 27'' = 3.71$ inches.

The area of the triangle $= \frac{3.71 \times 3}{2} = 5.56$ □".

The area of the segment $= 6.15 - 5.56 = 0.59$ □".
The area of the rectangle $ABGF = (15 - 3.71) \times 3 = 33.87$ □".
In the triangle CDE,

$$\tan C = \frac{5}{8\frac{3}{4}} = .57143 = \frac{DE}{CD} = \frac{.5}{CD}.$$

Hence, $$CD = \frac{.5}{.57143} = .875 \text{ inch.}$$

The area of the triangle $CDE = \frac{.875 \times .5}{2} = .22$ □", nearly.

$.22 \times 2 = .44$ □" = twice the area of the triangle CDE. Since, in the triangle RST, RS is perpendicular to CR and TS is perpendicular to CF, the angle $S =$ angle C; hence, tan $S = .57143 = \frac{RT}{SR} = \frac{RT}{.5}$; therefore, $RT = .57143 \times .5 = .29$ inch, nearly.

Area $RST = \frac{.29 \times .5}{2} = .07\,\square''$, nearly.

Twice the area of the triangle $RST = .07 \times 2 = .14\,\square''$.

Since tan $C = .57143$, $C = 29° 44' 42''$.

In the rectangle $DESR$, $DR = CT - (CD + RT)$. But $CT = \frac{5}{\sin 29° 44' 42''} = \frac{5}{.49614} = 10.08$ inches.

$CD + RT = .875 + .29 = 1.16$. $DR = 10.08 - 1.16 = 8.92$. $8.92 \div 2 = 4.46\,\square''$ = the area of $DESR$.

Twice the area of the rectangle $DESR = 4.46 \times 2 = 8.92\,\square''$.

The area of the rectangle $HIKL = 14 \times 1\frac{1}{2} = 21\,\square''$.

The area of the rectangle $LMNP = \left(\frac{14 - 11}{2}\right) \times 3 = 1\frac{1}{2} \times 3 = 4\frac{1}{2}\,\square''$; and $4\frac{1}{2} \times 2 = 9\,\square''$.

The area of the triangle $MON = \left(\frac{11 - 9}{2} \times 3\right) \div 2 = 1.5\,\square''$.

Twice the area of the triangle $MON = 1.5$ inches $\times 2 = 3\,\square''$.

Then, $50.27 + 33.87 + 0.44 + 0.14 + 8.92 + 21 + 9 + 3 = 126.64\,\square''$. $12.57 + 0.59 = 13.16\,\square''$. $126.64 - 13.16 = 113.48\,\square''$.

Therefore, the area of the figure $= 113.48\,\square''$. Ans.

THE MENSURATION OF SOLIDS.

146. A **solid,** or body, has three dimensions: length, breadth, and thickness. The sides which enclose it are called the **faces** and their intersections are called **edges.**

147. The **entire surface** of a solid is the area of the whole outside of the solid, including the ends.

148. The **convex surface** of a solid is the same as the entire surface, except that the areas of the ends are not included.

149. The **volume** of a solid is expressed by the number of times it will contain another volume, called the unit of volume. Instead of the word *volume*, the expression **cubical contents** is frequently used.

THE PRISM AND CYLINDER.

150. A **prism** is a solid whose ends are equal polygons and parallel to each other and whose sides are parallelograms.

151. A **parallelopipedon,** Fig. 89, is a prism whose bases (ends) are parallelograms.

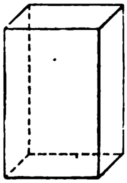

FIG. 89.

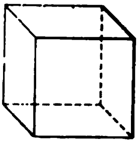

FIG. 90.

152. A **cube,** Fig. 90, is a parallelopipedon whose faces and ends are squares.

153. The cube, whose edges are equal to the unit of length, is taken as the unit of volume when finding the volume of a solid.

Thus, if the unit of length is 1 inch, the unit of volume will be the cube each of whose edges measures 1 inch, or 1 *cubic inch;* and the number of cubic inches the solid contains will be its volume. If the unit of length is 1 foot, the unit of volume will be 1 *cubic foot,* etc. Cubic inch, cubic foot, and cubic yard are abbreviated to cu. in., cu. ft., and cu. yd., respectively.

154. Prisms take their names from their bases. Thus, a *triangular prism* is one whose bases are triangles; a *pentagonal prism* is one whose bases are pentagons, etc.

155. A **cylinder,** Fig. 91, is a round body of uniform diameter with circles for its ends.

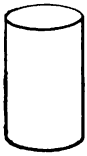

FIG. 91.

156. A **right prism,** or **right cylinder,** is one whose center line (axis) is perpendicular to its base. In this section all of the solids will be considered as having their center lines perpendicular to their bases.

157. The **altitude** of a prism or cylinder is the perpendicular distance between its two ends.

158. To find the area of the convex surface of any right prism, or right cylinder.

Rule.—*Multiply the perimeter of the base by the altitude.*

Let p be the perimeter of the base, h the altitude, and S the convex surface; then,

$$S = p\,h.$$

Example 1.—In a right prism whose base is a square, one side of which is 9 inches and whose altitude is 16 inches, what is its convex area?

Solution.— $9 \times 4 = 36 =$ the perimeter of the base. Applying the formula,

$S = p\,h = 36 \times 16 = 576$ □″, the convex area. Ans.

To find the entire area, add the areas of the two ends to the convex area:

Example 2.—What is the entire area of the parallelopipedon mentioned in the last question?

Solution.—The area of one end $= 9^2 = 81$ □″. $81 \times 2 = 162$ □″, or the area of both ends. $576 + 162 = 738$ □″, the entire area of the parallelopipedon. Ans.

Example 3.—What is the entire area of a right cylinder whose base is 16 inches in diameter and whose altitude is 24 inches?

Solution.— $16 \times 3.1416 = 50.27$ inches, or the perimeter (circumference) of the base. $50.27 \times 24 = 1{,}206.48$ □″, the convex area.
$16^2 \times .7854 \times 2 = 402.12$ □″, the area of the ends.
$1{,}206.48 + 402.12 = 1{,}608.6$ □″, the entire area. Ans.

159. To find the volume of a right prism, or cylinder:

Rule.—*The volume of any right prism or cylinder equals the area of the base multiplied by the altitude.*

Let A be the area of the base, h the altitude, and V the volume; then,

$$V = A\,h.$$

If the given prism is a cube, the three dimensions are all equal and the volume equals the cube of one of the edges. Hence, if the volume is given, the length of an edge is found by extracting the cube root.

If the volume and area are given, the altitude $= \frac{V}{A}$. If the cylinder or prism is hollow, the volume is equal to the area of the ring or base multiplied by the altitude.

EXAMPLE 1.—What is the volume of a rectangular prism whose base is 6 in. × 4 in. and whose altitude is 12 inches?

SOLUTION.—The base of a rectangular prism is a rectangle. Hence, $6 \times 4 = 24\,\square''$, the area of the base. Applying the formula $V = A\,h = 24 \times 12 = 288$ cu. in., or the volume. Ans.

EXAMPLE 2.—What is the volume of a cube whose edge is 9 inches?

SOLUTION.— $9^3 = 9 \times 9 \times 9 = 729$ cu. in., the volume. Ans.

EXAMPLE 3.—What is the volume of a cylinder whose base is 7 inches in diameter and whose altitude is 11 inches?

SOLUTION.— $7^2 \times .7854 = 38.48\,\square''$, the area of the base. Applying the formula, $V = A\,h = 38.48 \times 11 = 423.28$ cu. in., the volume. Ans.

THE PYRAMID AND CONE.

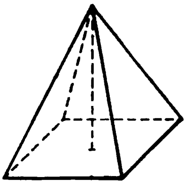

FIG. 92.

160. A **pyramid,** Fig. 92, is a solid whose base is a polygon and whose sides are triangles uniting at a common point called the **vertex.**

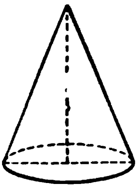

FIG. 93.

161. A **cone,** Fig. 93, is a solid whose base is a circle and whose convex surface tapers uniformly to a point called the **vertex.**

162. The **altitude** of a pyramid or cone is the perpendicular distance from the vertex to the base.

163. The **slant height** of a *pyramid* is a line drawn from the vertex perpendicular to one of the sides of the base. The slant height of a *cone* is any straight line drawn from the vertex to the circumference of the base.

164. To find the area of a right pyramid or right cone:

Rule.—*The convex area of a right pyramid or cone equals the perimeter of the base multiplied by one-half the slant height.*

Let p be the perimeter, s the slant height, and C the convex area; then,

$$C = \frac{ps}{2}.$$

EXAMPLE 1.—What is the convex area of a pentagonal pyramid, if each side of the base measures 6 inches and the slant height equals 14 inches?

SOLUTION.—The base of the pentagonal pyramid is a pentagon, and, consequently, it has five sides.

$6 \times 5 = 30$ inches, or the perimeter of the base.

Applying the formula,

$$C = \frac{ps}{2} = \frac{30 \times 14}{2} = 210 \,\square'', \text{ the convex area. Ans.}$$

EXAMPLE 2.—What is the entire area of a right cone whose slant height is 17 inches and whose base is 8 inches in diameter?

SOLUTION.— $8 \times 3.1416 = 25.1328$ inches, the perimeter.

$25.1328 \times \frac{17}{2} = 213.63 \,\square''$, the convex area.
$8^2 \times .7854 = 50.27 \,\square''$, the area of base.

Sum $= 263.90 \,\square''$, the entire area. Ans.

165. To find the volume of a right pyramid or cone:

Rule.—*The volume of a right pyramid or cone equals the area of the base multiplied by one-third of the altitude.*

Let A be the area of the base, h the altitude, and V the volume; then,

$$V = \frac{A\,h}{3}.$$

If the base of the pyramid is a regular polygon, its area may be found by the rules in Arts. **140** and **141.**

Example 1.—What is the volume of a triangular pyramid, the edges of whose base each measure 6 inches and whose altitude is 8 inches?

Solution.—The base is an equilateral triangle, hence, applying the rule in Art. **141,** the area is $6^2 \times .433 = 15.59 \,\square''$. Applying the formula,

$$V = \frac{A\,h}{3} = \frac{15.59 \times 8}{3} = 41.57 \text{ cu. in.} \quad \text{Ans.}$$

Example 2.—What is the volume of a cone whose altitude is 18 inches and whose base is 14 inches in diameter?

Solution.— $14^2 \times .7854 = 153.94 \,\square''$, the area of the base. Applying the formula,

$$V = \frac{A\,h}{3} = \frac{153.94 \times 18}{3} = 923.64 \text{ cu. in., the volume.} \quad \text{Ans.}$$

THE FRUSTUM OF A PYRAMID OR CONE.

166. If a pyramid be cut by a plane parallel to the base, as in Fig. 94, so as to form two parts, the lower part is called the **frustum** of the pyramid.

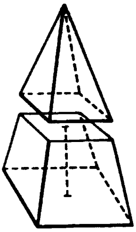

Fig. 94.

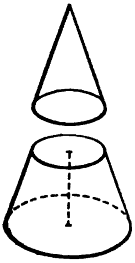

Fig. 95.

167. If a cone be cut in a similar manner, as in Fig. 95, the lower part is called the **frustum** of the cone.

168. The upper end of the frustum of a pyramid or cone is called the **upper base,** and the lower end the **lower base.** The **altitude** of a frustum is the perpendicular distance between the bases.

169. To find the convex area of a frustum of a right pyramid or right cone:

Rule.—*The convex area of a frustum of a right pyramid or right cone equals one-half the sum of the perimeters of its bases multiplied by the slant height of the frustum.*

Let p be the perimeter of the lower base, p' that of the upper base, s the slant height, and C the convex area; then,

$$C = \left(\frac{p+p'}{2}\right)s.$$

Example 1.—Given, the frustum of a triangular pyramid, in which each side of the lower base measures 10 inches, each side of the upper base measures 6 inches, and whose slant height is 9 inches, find the convex area.

Solution.— 10 inches × 3 = 30 inches, the perimeter of the lower base.

6 inches × 3 = 18 inches, the perimeter of the upper base.

Applying the formula,

$$C = \left(\frac{p+p'}{2}\right)s = \frac{30+18}{2} \times 9 = 216\ \square'', \text{ the convex area. Ans.}$$

Example 2.—If the diameters of the two bases of a frustum of a cone are 12 inches and 8 inches, respectively, and the slant height is 12 inches, what is the entire area of the frustum?

Solution.— $\frac{(12 \times 3.1416) + (8 \times 3.1416)}{2} \times 12 = 376.99\ \square''$, the area of the convex surface.

$$8^2 \times .7854 = 50.27\ \square''.$$
$$12^2 \times .7854 = 113.1\ \square''.$$

$113.1 + 50.27 = 163.37\ \square''$, the area of the two ends.

$376.99 + 163.37 = 540.36\ \square''$, the entire area of the frustum. Ans.

170. To find the volume of the frustum of a pyramid or cone:

Rule.—*Add the areas of the upper base, the lower base, and the square root of the product of the areas of the two bases; multiply this sum by one-third of the altitude.*

Let A be the area of lower base, a the area of upper base, h the altitude, and V the volume; then,

$$V = (A + a + \sqrt{Aa})\frac{h}{3}.$$

If the base is a regular polygon, the area may be found by the rules in Arts. **140** and **141.**

Example 1.—Given, a frustum of a hexagonal pyramid, each edge of the lower base measuring 8 inches and each edge of the upper base 5 inches, and whose altitude is 14 inches, what is its volume?

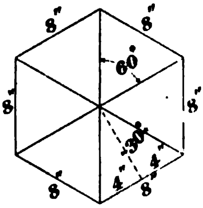

Fig. 96.

Solution.—A hexagonal pyramid is one whose base is a regular hexagon, as shown in Fig. 96. Hence, using the formula in Art. **140,**

$$A = \frac{n\,l^2 \cot x}{4} = \frac{6 \times 8 \times 8 \times 1.73205}{4} = 166.28\,\square''.$$

In a similar way, find the area of the upper base to be $64.95\,\square''$. Then, applying the formula,

$$166.28 + 64.95 + \sqrt{166.28 \times 64.95} = 166.28 + 64.95 + 103.92 = 335.15.$$

$$335.15 \times \tfrac{14}{3} = 1{,}564.03 \text{ cu. in.} = \text{the volume.} \quad \text{Ans.}$$

Example 2.—What is the volume of a frustum of a cone whose upper base is 8 inches, the lower base is 12 inches in diameter, and whose altitude is 15 inches?

Solution.—The area of the upper base is $8^2 \times .7854 = 50.27\,\square''$
The area of the lower base is $12^2 \times .7854 = 113.1\,\square''$, nearly.
The square root of their product is $\sqrt{50.27 \times 113.1} = 75.4$.

$$50.27 + 113.1 + 75.4 = 238.77.$$

$$238.77 \times \tfrac{15}{3} = 1{,}193.85 \text{ cu. in., the volume.} \quad \text{Ans.}$$

THE SPHERE.

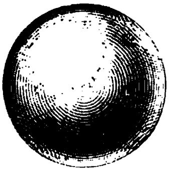
Fig. 97.

171. A **sphere,** Fig. 97, is a solid bounded by a uniformly curved surface, every point of which is equally distant from a point within, called the center.

The word **ball** is commonly used instead of sphere.

172. To find the area of the surface of a sphere:

Rule.—*The area of the surface of a sphere equals the square of the diameter multiplied by 3.1416.*

Let S be the surface and d the diameter; then,

$$S = \pi d^2.$$

EXAMPLE.—What is the area of the surface of a sphere whose diameter is 14 inches?

SOLUTION.—Applying the formula, $S = \pi d^2 = 3.1416 \times 14^2 = 3.1416 \times 14 \times 14 = 615.75 \square''$, the area. Ans.

173. To find the volume of a sphere:

Rule.—*The volume of a sphere equals the cube of the diameter multiplied by .5236.*

Let V be the volume and d the diameter; then,

$$V = \frac{1}{6} \pi d^3 = .5236\, d^3.$$

EXAMPLE.—What is the weight of a lead cannon ball 12 inches in diameter, a cubic inch of lead weighing .41 pound?

SOLUTION.—Applying the formula, $V = .5236\, d^3 = .5236 \times 12 \times 12 \times 12 = 904.78$ cubic inches, the volume of the ball. $904.78 \times .41 = 370.96$ lb. Ans.

The volume of a spherical shell or hollow sphere is equal to the difference in volume between two spheres having the outer and inner diameters of the shell.

174. To find the diameter of a sphere of known volume:

Rule.—*Divide the volume by .5236 and extract the cube root of the quotient. The result is the diameter.*

$$d = \sqrt[3]{\frac{V}{.5236}} = 1.2407 \sqrt[3]{V}.$$

EXAMPLE.—The volume of a sphere is 96.1 cubic inches. What is its diameter?

SOLUTION.—Applying the formula,

$$d = \sqrt[3]{\frac{V}{.5236}} = \sqrt[3]{\frac{96.1}{.5236}} = 1.2407 \sqrt[3]{96.1} = 5.68+ \text{ in. Ans.}$$

THE CYLINDRICAL RING.

175. If any solid be sliced in pieces whose adjacent surfaces are flat, any piece is called a **plane section** of the solid.

Plane sections are divided into three classes: Longitudinal sections, cross-sections, and right sections. A **longitudinal section** is any plane section taken lengthwise through the solid. Any other plane section is called a **cross-section.** If the surface exposed by taking a plane section of a solid is perpendicular to the center line of the solid, the section is called a **right section.** The surface exposed by any longitudinal section of a cylinder is a rectangle. The surface exposed by a right section of a cube is a square; of a cylinder or cone, a circle; an oblique cross-section of a cylinder is an ellipse. The lower half of a right section of a cone or pyramid is called a frustum of the cone or pyramid.

176. To find the convex area of a cylindrical ring:

A **cylindrical ring** is a cylinder bent to a circle. The **altitude** of the cylinder before bending is the same as the length of the dotted center line *D*, Fig. 98.

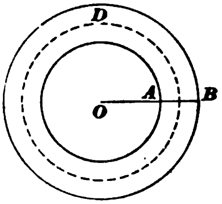

FIG. 98.

177. The **base** will correspond to a cross-section on the line *A B* drawn from the center *O*. Hence, to find the convex area, multiply the circumference of an imaginary cross-section on the line *A B* by the length of the center line *D*.

EXAMPLE.—A piece of round-iron rod is bent into circular form to make a ring for a chain; if the outside diameter of the ring is 12 inches and the inside diameter is 8 inches, what is its convex area?

SOLUTION.—The diameter of the center circle equals one-half the sum of the inside and outside diameters $= \frac{12 + 8}{2} = 10$, and $10 \times 3.1416 = 31.416$ inches, the length of the center line. The radius of the inside circle is 4 inches, of the outside circle 6 inches; therefore, the diameter of the cross-section on the line *A B* is 2 inches. Then, $2 \times 3.1416 = 6.2832$ inches, and $6.2832 \times 31.416 = 197.4$ □″, or the convex area. Ans.

178. To find the volume of a cylindrical ring:

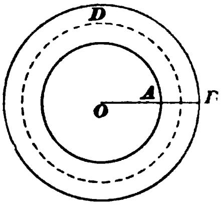

FIG. 99.

The volume will be the same as that of a cylinder whose altitude equals the length of the dotted center line D, Fig. 99, and whose base is the same as a cross-section of the ring on the line A B, drawn from the center O. Hence, to find the volume of a cylindrical ring, multiply the area of an imaginary cross-section on a line A B by the length of the center line D.

EXAMPLE.—What is the volume of a cylindrical ring whose outside diameter is 12 inches and whose inside diameter is 8 inches?

SOLUTION.—The diameter of the center circle equals one-half the sum of the inside and outside diameters $= \frac{12 + 8}{2} = 10$.

$10 \times 3.1416 = 31.416$ inches, the length of the center line.

The radius of the outside circle = 6 inches, of the inside circle = 4 inches; therefore, the diameter of the cross-section on the line *A B* = 2 inches.

Then, $2^2 \times .7854 = 3.1416$ □″, the area of the imaginary cross-section.

And $3.1416 \times 31.416 = 98.7$ cu. in., the volume. Ans.

EXAMPLES FOR PRACTICE.

1. Find the weight of an iron bar 16 feet long and 2 inches in diameter, the weight of iron being taken at .28 pound per cubic inch. Ans. 168.89 lb.

2. What is the area of the entire surface of a hexagonal prism 12 inches long, each edge of the base being 1 inch long? Ans. 77.196 sq. in.

3. What is the volume of a triangular pyramid, one edge of whose base measures 3 inches and whose altitude is 4 inches? Ans. 5.2 cu. in.

4. Find the volume of a cone whose altitude is 12 inches and the circumference of whose base is 31.416 inches. Ans. 314.16 cu. in.

5. A round tank is 8 feet in diameter at the top (inside) and 10 feet at the bottom. If the tank is 12 feet deep, how many gallons will it hold, there being 231 cubic inches in a gallon? Ans. 5,734.2 gal.

6. Required, the area of the convex surface of the frustum of a square pyramid whose altitude is 16 inches, one side of the lower base being 28 inches long and of the upper base 10 inches. Ans. 1,395.18 sq. in.

7. What is the volume of a sphere 30 inches in diameter? Ans. 14,137.2 cu. in

8. How many square inches in the surface of the sphere of example 7? Ans. 2,827.44 sq. in.

9. Required, the area of the convex surface of a circular ring, the outside diameter of the ring being 10 inches and the inside diameter 7¼ inches. Ans. 107.95 sq. in.

10. Find the cubical contents of the ring in the last example. Ans. 33.734 cu. in.

11. The volume of a sphere is 606.132 cubic inches; required, the area of the convex surface of a cone whose slant height is 10 inches and the diameter of whose base is the same as the diameter of the sphere. Ans. 164.934 sq. in.

12. What is the volume of the frustum of example 6? Ans. 6,208 cu. in.

PROJECTIONS.

179. If perpendiculars be drawn from the extremities of a line, as $A\ B$, Fig. 100, or Fig. 102, to another line, as $H\ K$, as shown in the figures, that portion of $H\ K$ included between the foot of each perpendicular is called the **projection** of $A\ B$ upon $H\ K$. Thus, $C\ D$ is the projection of $A\ B$ upon $H\ K$, the point C is the projection of the point A upon $H\ K$, and the point D is the projection of the point B upon $H\ K$.

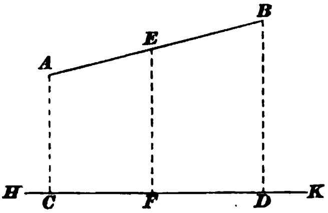

FIG. 100.

The projection of any point of $A\ B$, as E, can be found by drawing a perpendicular from E to $H\ K$, and the point where this perpendicular intersects $H\ K$ is its projection; in this case the point F is the projection of the point E upon $H\ K$.

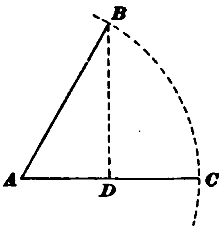

FIG. 101.

From the foregoing it is evident that the projection of any straight line upon another line is found by considering the inclined line as the hypotenuse of a right triangle, as $A\ B$, Fig. 101, so that the projected length may be found by multiplying the hypotenuse by the cosine of the angle that it makes with the other line; thus, $A D$ is the projection of $A B$ upon the horizontal line $A C$ and $B D$ is its projection on a vertical line.

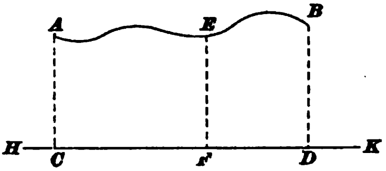

FIG. 102.

It makes no difference whether a line is straight or curved, the method of finding the projection is exactly the same.

In a similar way, a surface is projected upon a flat surface.

Thus, it is desired to project the irregular surface $a\ b\ d\ c$, Fig. 103, upon the flat surface $A B D C$. Draw the lines $a a'$, $b b'$ perpendicular to the flat surface; join the points a' and b' where these perpendiculars intersect the flat surface $A B D C$ by a straight line $a' b'$, and $a' b'$ is the projection of $a\ b$ upon $A B D C$. The projection of the surface $a b d c$ upon the plane $A B D C$ is in this case the quadrilateral $a' b' d' c'$.

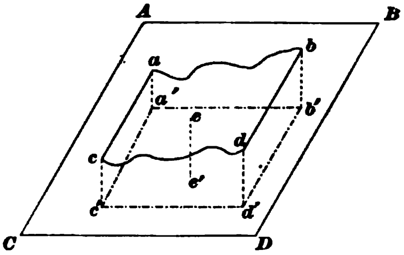

FIG. 103.

SYMMETRICAL AND SIMILAR FIGURES.

180. An **axis of symmetry** is any line so drawn that, if the part of the figure on one side of the line be folded on this line, it will coincide exactly with the other part, point for point and line for line. Thus, in Fig. 104, if the upper half be folded over on the diameter $C D$, it will coincide exactly with the lower half; also, if the part on the right of the diameter $A B$ be folded over on $A B$, it will coincide exactly with the part on the left of this line.

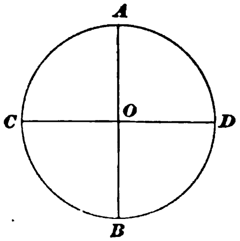

FIG. 104.

It is evident from the above that a circle may have any number of axes of symmetry. In certain cases, however, a figure may be symmetrical with regard to only one axis. Thus, the isosceles triangle *A B C*, Fig. 105, is symmetrical with regard to the axis *B D*, because the part *B C D* would coincide with the part *B A D* if folded over on the line *B D*; but no other axis of symmetry could be drawn. A rectangle has two axes of symmetry at right angles to each other. A hexagon has six axes of symmetry.

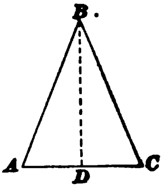

FIG. 105.

181. Similar figures are those which are alike in form. As in the case of triangles, which have been considered, two figures, to be similar, must have their corresponding sides in proportion, and the angles of one equal to the corresponding angles of the other. Any two circles or any two regular polygons of the same number of sides are similar.

182. The areas of two similar figures are to each other as the squares of any one dimension. Thus, a parallelogram 10 inches long and 4 inches wide contains 40 square inches. A similar parallelogram 20 inches long would be 8 inches wide and would contain 160 square inches, while the two areas would be to each other as the squares of the corresponding sides of the parallelograms. That is,

$$40 : 160 = 10^2 : 20^2,$$

or

$$40 : 160 = 4^2 : 8^2.$$

EXAMPLE.—A circle 10 inches in diameter contains 78.54 square inches; what is the area of one 12 inches in diameter?

SOLUTION.—Let $x =$ the area of the larger circle. Then,

$$78.54 : x = 10^2 : 12^2, \text{ or } x = \frac{78.54 \times 144}{100} = 113.0976 \text{ sq. in. Ans.}$$

183. The cubical contents (and weights) of **similar solids** are to each other as the *cubes* of any one dimension.

EXAMPLE 1.—If a cast-iron ball 9 inches in diameter weighs 100 pounds, what would one 15 inches in diameter weigh?

SOLUTION.— $100 : x = 9^3 : 15^3$,

or $x = \frac{100 \times 3,375}{729} = 462.96$ lb., the weight of larger ball. Ans.

EXAMPLE 2.—A regular hexagon has sides 5 inches long; how much greater will the area of another regular hexagon be whose sides are 30 inches long?

SOLUTION.— $30 \div 5 = 6$, or the length of a side of a 30-inch hexagon is 6 times as great as the length of a side of a 5-inch hexagon; the area will be $6^2 = 36$ times as great. Ans.

This example may also be solved by letting 1 represent the area of the 5-inch hexagon. Then,

$$1 : x = 5^2 : 30^2, \text{ or } x = \frac{900}{25} = 36.$$

184. The principles given in Arts. **182** and **183** are extremely useful and find many applications in practice, especially in drafting-room practice. Draftsmen almost invariably make their drawings to scale, as it is termed; that is, the size of the paper they are using prevents them from drawing the machine or other object full size, and they are obliged to draw them one-half size, one-quarter size, one-twelfth size, etc.; in other words, each line or dimension on the drawing is $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{12}$, etc. the length of the corresponding line or dimension on the object. For example, the object represented in Fig. 88 is only $\frac{1}{8}$ the actual size, i. e., the length of each line or dimension in the cut is only $\frac{1}{8}$ as long as it would be were the drawing made full size. Suppose there were no dimensions given, but we knew that the drawing was $\frac{1}{8}$ the actual size, and we wanted to know the actual area of the figure. We could measure such lines and dimensions as were necessary and calculate the area of the figure as represented on the drawing. Then, knowing that this figure is *similar* in outline to the object itself, and that any line or dimension on the object is 8 times as long as the corresponding line or dimension on the drawing, we could find the area of the object by multiplying the area of figure by 8^2, or 64. The multiplier 8^2 is obtained from the proportion (see Art. **182**),

$$\text{area of figure} : \text{actual area of object} = 1^2 : 8^2,$$

or

$$\text{actual area of object} = 64 \times \text{area of figure},$$

From the foregoing, it will be readily seen that if we know the area of any figure, no matter what its shape, the area of any similar figure may be found by finding the ratio of any two lines or dimensions similarly placed and *squaring the ratio.* Also, if the volume of any solid is known, the volume of a similar solid may be found by finding the ratio of any two lines or dimensions similarly placed and *cubing the ratio.* For example, suppose the area of a certain figure is known to be 1,024 square inches and it is desired to find the area of a similar figure, the ratio of any two corresponding dimensions being 5 : 4 or $1\frac{1}{4} : 1$. The area desired is obtained by multiplying the known area by $(1\frac{1}{4})^2$, or by squaring the ratio 5 : 4, obtaining 25 : 16; putting this in the fractional form $\frac{25}{16}$, and multiplying 1,024 by this fraction, we get $1,024 \times \frac{25}{16} = 1,600$ square inches.

Again, if the volume of a certain solid is known, the volume of a similar solid that is, say, with dimensions $\frac{1}{3}$ as large, may be readily found by multiplying the known volume by $(\frac{1}{3})^3 = \frac{1}{27}$.

SURVEYING.

GEOMETRY.

1180. If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, the triangles are equal in all their parts. Thus, in the two triangles *A B C* and *D E F*, Fig. 236, if the side *A B* is equal to the side *D E*; the side *B C* to the side *E F*, and the angle *B* to the angle *E*, the triangles are equal in every respect.

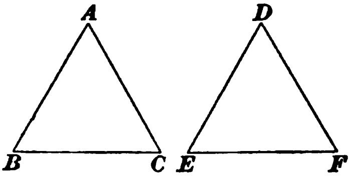

FIG. 236.

1181. If a straight line, *A B*, Fig. 237, intersects two parallel straight lines, *C D* and *E F*, it is called a **secant** with respect to them, and the eight angles formed about the points of intersection have different names applied to them with respect to each other, as follows:

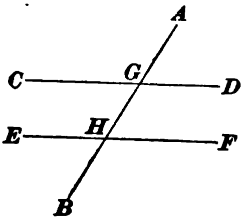

FIG. 237.

First—**Interior angles on the same side** are those which lie on the same side of the secant and *within* the other two lines. Thus, in Fig. 237, *H G D* and *G H F* are interior angles on the same side.

Second—**Exterior angles on the same side** are those which lie on the same side of the secant but *without* the other two lines. Thus, *A G D* and *F H B* are exterior angles on the same side.

Third—**Alternate interior angles** are those which lie on opposite sides of the secant and *within* the other two lines. Thus, *C G H* and *G H F* are alternate interior angles.

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Fourth—**Alternate exterior angles** are those which lie on opposite sides of the secant and *without* the other two lines. Thus, AGC and FHB are alternate exterior angles.

Fifth—**Opposite exterior and interior angles** are those which lie on the same side of the secant, the one *within* and the other *without* the other two lines. Thus, AGD and GHF are opposite exterior and interior angles.

1182. If a straight line intersects two parallel lines, the sum of the *interior* angles on the same side is equal to two right angles, and the sum of the *exterior* angles on the same side is also equal to two right angles. Thus, in Fig. 237, the interior angles DGH and FHG are together equal to two right angles, and the exterior angles DGA and FHB are together equal to two right angles.

1183. If a line intersects two parallel straight lines, the alternate interior angles are equal to each other, and the alternate exterior angles are also equal to each other. Thus, in Fig. 237, the angle CGH is equal to FHG, and angle CGA is equal to FHB.

1184. The **complement** of an angle is the difference between that angle and a right angle. Thus, in Fig. 238, ABE is the complement of DBE.

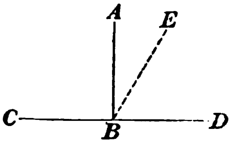


FIG. 238.

1185. The **supplement** of an angle is the difference between that angle and two right angles. Thus, CBE is the supplement of DBE.

1186. In any triangle, a line drawn parallel to one of the sides divides the other sides proportionally. Thus, in the triangle ABC, Fig. 239, the line DE drawn parallel to BC divides the sides AB and AC proportionally; that is,

$$AB : AD :: AC : AE;$$
$$AD : DB :: AE : EC, \text{ and}$$
$$AB : DB :: AC : EC.$$

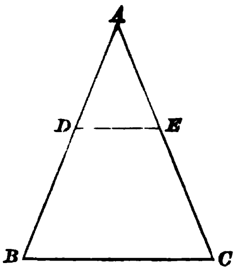


FIG. 239.

1187. Polygons are **similar** when they are mutually equiangular and have their homologous sides proportional.

In similar polygons, any points, lines, or angles similarly situated in each are called **homologous.** The ratio of a side of one polygon to its homologous side in another is called the **ratio of similitude** of the polygons.

1188. Triangles which are mutually equiangular are similar, and their areas are to each other as the squares of their homologous sides.

Thus, in the triangles $A B C$ and $D E F$, Fig. 240, if the angle A is equal to the angle D; the angle B to the angle E, and the angle C to the angle F, the triangles are similar, and their areas are to each other as the squares of their homologous sides.

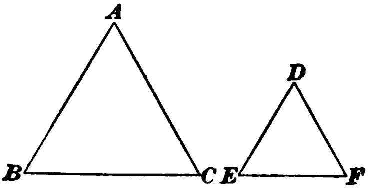


FIG. 240.

For example, if $B C = 80$ feet, $E F = 50$ feet, and the area of the triangle $A B C = 1{,}600$ sq. ft., then

$$80^2 : 50^2 :: 1{,}600 : \text{area of } D E F, \text{ or}$$

$$6{,}400 : 2{,}500 :: 1{,}600 : 625 \text{ sq. ft.}$$

Hence, area of $D E F$ is 625 sq. ft.

1189. The areas of *similar* polygons are to each other as the squares of their homologous sides.

Thus, if the area of a regular hexagon with a side of 10 inches is 259.809 sq. in., the area of a similar hexagon whose side is 15 inches may be found as follows:

$$10^2 : 15^2 :: 259.809 : \text{area required, or}$$

$$100 : 225 :: 259.809 : 584.57 \text{ sq. in.}$$

1190. The circumferences of circles are to each other as their diameters, and their areas are to each other as the squares of their diameters.

Thus, if the circumference of a circle 12 inches in diameter is 37.7 inches, the circumference of a circle of 18 inches diameter may be found by proportion. Thus,

$12 : 18 :: 37.7 : 56.55$ in., the circumference required.

Again, if the area of a circle of 12 inches diameter is 113.098 sq. in., the area of a circle of 18 inches diameter may be found as follows:

$12^2 : 18^2 :: 113.098$: area required, or

$144 : 324 :: 113.098 : 254.47$ sq. in.

1191. An angle formed by a tangent and a chord meeting at the point of contact is measured by half the included arc.

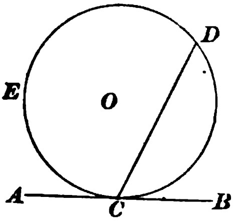


FIG. 241.

Thus, in Fig. 241, the angle *A C D* formed by the meeting of the tangent *A B* and the chord *C D* is measured by half the arc *C E D*. Similarly, the angle *B C D* is measured by half the arc *C D*.

1192. Two tangents to a circle drawn from any point are equal, and if a chord be drawn joining these tangent points, the angles between the chord and the tangents are equal.

Thus, in Fig. 242, the two tangents *A B* and *A C* drawn to the circle from the point *A* are equal, and the angles *A B C* and *A C B*, formed by the chord and tangents, are equal to each other.

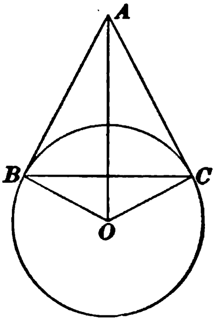


FIG. 242.

1193. In the same or equal circles equal chords subtend equal angles at the center and also at the circumference.

Thus, in Fig. 243, the angles *A O B*, *B O C*, and *C O D* subtended by the equal chords *A B*, *B C*, and *C D* are equal to each other.

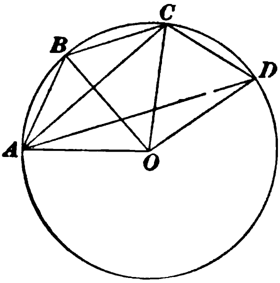


FIG. 243.

Again, the angles *B A C* and *C A D* are also equal to each other.

1194. In Fig. 244, let *A B C* be any triangle. If one of the sides, as *A C*, is prolonged, the angle *B C D* included between the side thus prolonged and the other side $B\ C$ of the triangle, which meets $A\ C$ at C, is called an **exterior angle.** The two remaining angles A and B of the triangle, which are opposite to the angle C, are called **opposite interior angles.** In any triangle, an exterior angle is equal to the sum of the two opposite interior angles; that is, in the above figure, the exterior angle $B\ C\ D$ is equal to the sum of the two opposite interior angles, A and B.

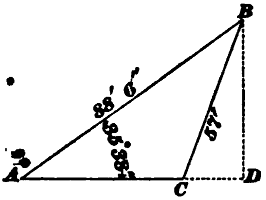


FIG. 244.

1195. PROBLEM.—Having given one of the angles of a triangle, one of the including sides, and the difference of the other two sides, to construct it.

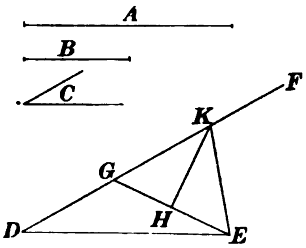


FIG. 245.

Let C, Fig. 245, be the given angle, A the given side, and B the difference of the other sides. Draw $D\ E$ equal to the given side A; at D make the angle $E\ D\ F$ equal to the given angle C; on $D\ F$ lay off $D\ G$ equal to the given difference B. Join $E\ G$. At the middle point H of $E\ G$ erect a perpendicular cutting $D\ F$ in K. Draw $K\ E$. $D\ E\ K$ is the required triangle.

COMPASS SURVEYING.

1196. The Compass.—The **surveyor's compass** consists of the magnetic needle, the case in which it is enclosed, and the support on which it is placed when ready for use.

1197. The Magnetic Needle.—The **magnetic needle** is a slender bar of steel, five or six inches in length, strongly magnetized, and mounted upon a finely pointed pivot on which it freely turns, always pointing in the same direction, viz.: the north and south line, or, as it is called, the **magnetic meridan.**

1198. North and South Ends of Needle.—Owing to the earth's attraction, the north end of the needle **dips,** that is, it is drawn downward from a horizontal position, while the south end is correspondingly raised. To prevent this dipping, several coils of platinum wire are wound around the south end of the needle (see Fig. 246), keeping it perfectly balanced upon its pivot and permitting entire freedom of movement. These coils of wire at once indicate to the observer which is the north end and which is the south end of the needle.

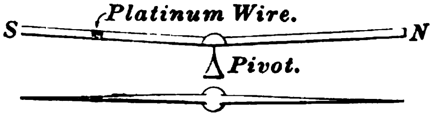


FIG. 246.

1199. The Sights.—At either end of a line passing through the needle pivot is a **sight,** which consists of an upright bar of brass, *A* and *B*. (See Fig. 247.) Narrow vertical slits, with holes at their top and bottom, divide this bar, as shown at *C* and *D*. These arrangements enable the observer to train the line of sight upon any desired object.

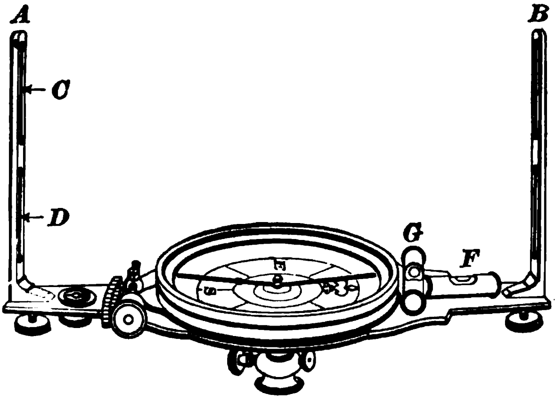


FIG. 247.

1200. The Divided Circle.—The compass box contains a graduated circle divided to half degrees, at the center of which is the pivot supporting the needle. The degrees are numbered from 0° to 90° both ways from the points where a line drawn through the slits would cut the circle.

1201. Lettering.—The lettering of the surveyor's compass is at first confusing to those learning its use. A person standing with his back to the south and facing the north will have the east on his right hand and the west on his left. These latter directions, viz., the east and the west, are reversed in the lettering of the compass. The reasons for this apparent error are explained in the following figures:

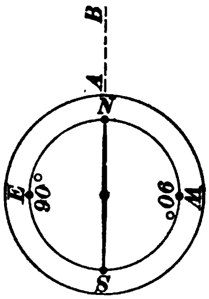


FIG 248.

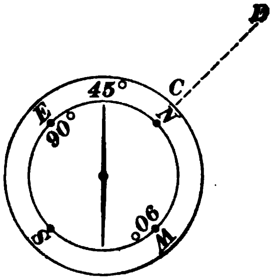


FIG. 249.

Suppose the needle and compass are pointing due north and south in the direction of the line *A B*, as shown in Fig. 248, and the line of survey changes its direction 45° to the right, or east. The magnetic needle will remain motionless, while the sights and the circle to which they are fastened will move until the sights point in the direction *C D*, Fig. 249, and, as the north end of the compass is ahead, the *needle* will read N 45° E, which is the *true direction* being run. If, however, the east and west points of the compass were the actual magnetic directions, i. e., the right hand east and the left hand west, the direction of the line *C D* would have read N 45° W, which would be the **reverse of** the actual direction.

1202. Levels.—On the compass plate are two small spirit levels *F* and *G*. (See Fig. 247.) They consist of glass tubes, curved slightly upwards and nearly filled with alcohol, leaving a small bubble of air in them. One of these tubes, *F*, is in the line of sight, the other, *G*, is at right angles to it. Their object is to enable the observer to place the compass in a perfectly horizontal position. This is done by so moving the compass as to bring the air bubbles to the centers of the tubes. To prove these bubbles to be in adjustment, proceed as follows: Having brought the bubbles to the centers of the tubes, revolve the compass through 180° or one-half of an entire revolution. If the bubbles remain in the centers of the tubes, they are in adjustment. If they do not so remain, bring them half way back to the middle of the tubes by means of small screws attached to the tubes, and the remainder of the way by moving the plate in the ordinary way, repeating the operation until the bubbles remain in the center of the tubes in every position of the compass.

1203. The Tripod.—The compass is usually supported by a single standard, shod with steel, and called a **Jacob's Staff.** A more perfect support, called a **tripod,** consists of three legs shod with steel and connected at the top so as to move freely. Both Jacob's Staff and tripod are connected with the compass by means of a ball and socket joint, which permits free movement in all directions.

1204. Defects of the Compass.—The compass is not intended for work requiring great accuracy. The direction to which the needle points can not be read with precision, and the perfect freedom of movement of the needle may be prevented by local attraction or by particles of dust adhering to the pivot. An inaccuracy of one-quarter of a degree in reading an angle, i. e., the amount of change in the direction of two lines, will cause them to separate from each other 1¾ feet in a distance of 400 feet.

Suppose the line $A B$, Fig. 250, is due east and west, and the line $B C$, which is an actual boundary, has a true direction of N 85° E, and suppose the surveyor reads the directions $B C$ as N 84° 45′ E. Let $B C = 400$ feet, then, the point C, when mapped, will take the position C', which is 1¾ feet to the left of C where it should be. Another defect of the compass lies in the fact that the magnetic needle does not always point in the same direction. This direction sometimes changes between sunrise and noon to the amount of one-quarter of a degree. Frequently its direction is changed by *local influence.* A piece of iron on the surface of the ground or a mass of iron ore beneath are frequent disturbing influences.

FIG. 250.

1205. Taking Bearings.—The **bearing** of a line is the angle which it makes with the direction of the magnetic needle. By the **course** of a line we mean its length and its bearing taken together. To take the bearing of a line, set the compass directly over a point of it, at one extremity, if possible. This may be done by means of a plumb bob suspended from the compass, or, if the compass be mounted on a Jacob's Staff, by firmly planting the staff directly on the line. Then, by means of the air bubbles, bring the compass to a perfectly level position. Let a flagman hold a rod carefully plumbed at another point of the line, preferably the other extremity of it, if he can be distinctly seen. Direct the sights upon this rod and as near the bottom of it as possible. Always keep the same end of the compass ahead; the north end is preferable, as it is readily distinguished by some conspicuous mark, usually a "*fleur de lis*," and always read the same end of the needle, that is, the north end of the needle if the north point of the compass is ahead, and *vice versa.* Before reading the angle, see that the eye is in the direct line of the needle so as to avoid error which would otherwise result from **parallax,** or apparent change of the position of the needle, due to looking at it obliquely.

The angle is read and recorded by noting, *first,* whether the *N* or *S* point of the compass is nearest the end of the needle being read; *second,* the number of degrees to which it points, and *third,* the letter E or W nearest the end of the needle being read.

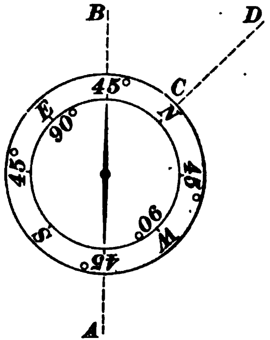


FIG. 251.

Let *A B,* in Fig. 251, be the direction of the magnetic needle, *B* being at the north end. Let the sights of the compass be directed along the line *C D.* The north point of the compass will be seen to be nearest the north end of the needle which is to be read. The needle which has remained stationary while the sights were being turned to *C D,* now points to 45° between the N and E points, and the angle is read north forty-five degrees east (N 45° E).

1206. Backsights.—A sure test of the accuracy of a bearing is to set up the compass at the other end of the line, i. e., the end first sighted to, and sight to a rod set up at the starting point. This process is called **backsighting.** If the second bearing is the same as the first, the reading is correct. If it is not the same, it shows that there is some disturbing influence at either one or the other end of the line. To determine which of these two bearings is the true one, the compass must be set up at one or more intermediate points, when two or more similar bearings will prove the true one. When a line can not be prolonged by magnetic bearings, on account of local attraction, the true direction is maintained by backsighting.

1207. Declination of the Needle.—The **magnetic meridian** is the direction of the magnetic needle. The **true meridian** is a true north and south line, which, if produced, would pass through the poles of the earth. The **declination of the needle** is the angle which the magnetic meridian and the true meridian make with each other.

In Fig. 252, let N S be the true meridian for any given place, and N^1 S^1 the magnetic meridian. The angle $N A N^1$ is the declination of the needle for that place.

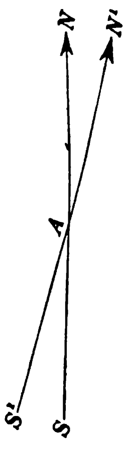


FIG. 252.

1208. The Polar Star.—There is a star in the northern hemisphere known as the North Star or Polaris. It is the extreme star in the row or line of stars forming what is commonly called the handle of the "Little Dipper." This star very nearly coincides with the true north point or pole, being removed only $1\frac{1}{2}^\circ$ from it. It revolves about the true pole, and twice in each revolution it is exactly in the true meridian; that is, in a vertical plane passing through the true pole P. See Fig. 253. One may know when the North Star is in the true meridian from the position of another star. This other star is in the handle of the "Big Dipper," or Ursa Major, the one nearest the bowl of the dipper, and is called Alioth. When the North Star is in the true meridian, Alioth will be found directly below it.

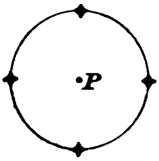


FIG. 253.

TO DETERMINE A TRUE MERIDIAN.

1209. By Observations of the North Star.—The time at which the North Star passes the meridian above the pole for every tenth day of the year is given in published tables, but those occurring in the day time are, of course, of no value with ordinary instruments. The following dates are available in almost every latitude of the United States:

TIME OF NORTH STAR PASSING THE MERIDIAN.

Months.	1st Day.	11th Day.	21st Day.
January.......	6:30 P. M.	5:51 P. M.	5:11 P. M.
August.......	4:33 A. M.	3:53 A. M.	3:14 A. M.
September.....	2:31 A. M.	1:52 A. M.	1:12 A. M.
October.......	12:34 A. M	11:50 P. M.	11:11 P. M.
November.....	10:28 P. M.	9:48 P. M.	9:09 P. M.
December.....	8:30 P. M.	7:50 P. M.	7:11 P. M.

Note from the table the time of passing the meridian, and, also, that it is the upper transit, i. e., above the pole. Select a suitable spot for permanently establishing the meridian line, and set up the transit and sight to Polaris, following it by moving the cross-hairs with the tangent screw. When it is exactly in line with Alioth, the line of sight will be in the true meridian. Points should be fixed immediately, a lamp being used to illuminate the cross-hairs.

1210. Changes in Magnetic Declination.—The magnetic declination is not fixed for any place, but constantly varies, its variations, however, being confined within fixed limits.

1211. To Correct Magnetic Bearings.—The declination at any place being known, the magnetic bearings may readily be reduced to true bearings.

In the Northeastern States, the declination is west; in the Western and Southern States, it is east; hence, the true bearing of a line in a Northeastern State, whose magnetic bearing is N W or S E, will be the sum of the magnetic bearing and the declination. If the magnetic bearing is N E or S W, the true bearing will be the difference of the magnetic bearing and the declination.

EXAMPLES FOR PRACTICE.

1212. Supposing the declination to be 7° west, what will be the true bearings of the following lines :

Magnetic Bearing.		True Bearings.
(1) N 12° 10′ W ?		(1) N 19° 10′ W.
(2) N 50° 15′ W ?		(2) N 57° 15′ W.
(3) S 11° 15′ E ?		(3) S 18° 15′ E.
(4) S 38° 10′ E ?		(4) S 45° 10′ E.
(5) N 50° 20′ E ?	Ans.	(5) N 43° 20′ E.
(6) S 20° 25′ W ?		(6) S 13° 25′ W.
(7) N 87° 30′ W ?		(7) S 85° 30′ W.
(8) N 5° 10′ E ?		(8) N 1° 50′ W.
(9) S 89° 20′ E ?		(9) N 83° 40′ E.
(10) S 3° 10′ W ?		(10) S 3° 50′ E.

1213. By Equal Shadows of the Sun.—On the south side of any level surface set up a flag-pole and plumb it with a plumb bob. Its horizontal projection will be a point as *S* in Fig. 254. Two or three hours before noon mark the point *A*, which is the extremity of the shadow cast by the flag-pole. Then, describe an arc *A B* with a radius equal to *S A*, the distance from *S* to the extremity of the shadow. After noon, note the moment when the shadow of the flag-pole touches another point of the arc, as *B*. Bisect the arc *A B* at *N*. The line *S N* is a true meridian.

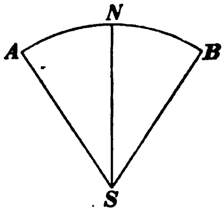


FIG. 254.

FIELD WORK.

1214. The Engineer's Chain.—The engineer's chain is one hundred feet in length, and is composed of one hundred links of steel wire, each one foot in length. Both ends of the chain are fitted with brass handles with swivel movements, and fitted with nuts for taking up any excess in length resulting from continual stretching. At each interval of ten feet is a brass tag with tally points to indicate its distance from the nearest end of the chain. Each tally point counts ten feet. At the middle point of the chain, the tag is of oval form to prevent confusion in reading the chain.

1215. Danger of Error.—There is much greater danger of error in reading the chain than in reading bearings. The danger arises from the fact that the compassman is usually one of experience, who knows the liability of error, and hence the necessity for care, while chainmen are often inexperienced, and, unfortunately, often careless.

1216. Keeping Chainmen in Line.—When the direction of a line has been given by setting up a flag, it becomes the business of the hind chainman or follower to keep the measurement on a straight line. The head chainman carries a flag which he moves to right or left, at the direction of the hind chainman, until it is in range with the flag towards which the compass is sighted, and this process is repeated at each chain measurement.

In railroad surveying, the line is divided into stations, which are one hundred feet or one chain apart. At each station a stake is driven and marked with a number corresponding to the number of chains which the station is distant from the starting point, which is numbered 0. When the end of a course falls between regular stations, it is called a sub-station, and the stake is marked by the number of the immediately preceding station plus the number of feet from it to the end of the course.

The line *A B*, in Fig. 255, is 650 feet in length. The starting point *A* is numbered *0*; each chain or one hundred feet is marked by a stake with numbers in regular notation. The point *B*, which is fifty feet from station *6*, is marked *6 + 50*.

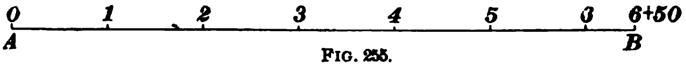


FIG. 255.

1217.—The Compass in Railroad Surveys.—The compass is of great value in running preliminary railroad lines, where local attraction is absent or very slight. The numerous delays encountered when running by backsights, as in transit work, where all obstacles to the line of sight must be cleared, are largely avoided in the use of the compass. The directions of all lines are referred to the magnetic needle, and, in case of an obstruction, such as a tree or a mass of rock, the compass can be quickly moved to the opposite side of the obstacle and the line continued without delay. In case the line produced is a foot or two off the true one, *it is a parallel to it*, and the error is not to be regarded as affecting the accuracy of *preliminary* information.

In the case of transit work, an error in the reading of an angle is a *cumulative* one, and practically destroys the value of the work. In the early days of railroad building, some lines were surveyed and built with the aid of the compass alone, but in America all location and construction depend for their precision upon the transit.

1218. Organization of Party.—A well-organized compass party consists of a chief of party, compassman, two chainmen, one flagman, two or more axmen, if the country be thickly wooded, and one stakeman. If possible, provide stakes of light, well-seasoned wood. For preliminary lines where stakes do no permanent service, pine is best. A convenient size is two feet six inches in length by two inches in width and half an inch in thickness. A strong, active stakeman will carry one hundred of these stakes, besides the ax with which to drive them. Provide both chainmen with marking crayons. The best crayon is of red chalk or German kiel. They are bought in a crude state, but a little work will shape them. They make a deep red mark, which will stand exposure for years. Require chainmen to be always provided with crayons. Instances of their forgetfulness too often occur. Require axmen to keep axes sharp. A dull ax is little better than no ax. Check length of chain with standard steel tape, lengthening or shortening by means explained in Art. **1214.** See that the compass is in perfect adjustment. If the line to be surveyed is of considerable length, a team of horses and driver with a strong spring wagon should be a part of the outfit.

1219. Actual Work.—The party is now prepared to move. The compassman sets up the compass at the starting point, which is marked 0. The chief of party goes ahead with the flagman, who carries a rod called a **flag.** This rod is from eight to twelve feet in length, and is divided into alternate red and white bands, each one foot in length. The flagman sets this flag up at the direction of the chief of party, the compassman sights the instrument to it, and the chainmen commence measuring the distance. The head chainman marks the stakes, and should always keep at least ten stakes marked ahead so as to avoid delay while measuring, and to insure consecutive numbering. Of these he need carry but five, leaving the remaining five with the stakeman. He must also carry a flag eight feet in length, and painted like the one carried by the flagman; this flag is used for "ranging in." As soon as the line is indicated by the head flag, the axmen should fall to work clearing whatever obstacles lie in the way of rapid chaining. By a little attention on their own part and occasional direction from the chainmen, they can keep well on line. At each station, and the moment the hind chainman has put the head chainman in line, the former should carefully note the number of the station at which he stands and call the number to the head chainman, who must answer by repeating the number next in notation. Thus, if the hind chainman stands at Station 25 he must call "Station 25," and the head chainman must reply "Station 26." The chainmen must be required to hold the chain "taut" while measuring, and in as nearly horizontal a position as possible. When the line of measurement rises or falls abruptly, the chainmen must "break the chain," as it is called. The best method of breaking the chain is shown in Fig. 256.

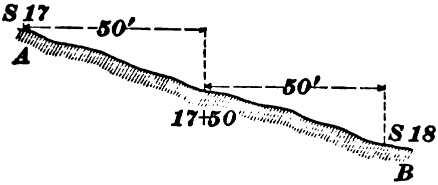


FIG. 256.

Let *A B* be a sloping surface lying in the line of measurement. The point *A* is at Station 17. Stretch out the chain to its full length and in proper line. The hind chainman will be at Station 17. The head chainman here takes the chain at the 50-foot tag and raises it until it is practically level. The flag he carries for ranging in will serve for a plumb line to mark the 50-foot point on the ground. The hind chainman then calls the number of his station, 17, the head chainman replying 17 + 50. The former then advances to 17 + 50 and holds the middle tag at the point marked by the rod. The head chainman then advances to the other end of the chain and repeats the operation, reaching Station 18. When the slope is steep, the chain must be broken into smaller sections. It is good practice for the flagman to carry, besides his flag, a number of light stakes at least eight feet in length and some strips of red flannel for **targets.** If the view for the compass is open, as soon as the compass is sighted and the flagman has a signal to that effect, he should replace the flag by one of the stakes with a piece of flannel attached and join the chief of party, who, unless the line is to be produced, has gone ahead to select another point for the flag. As soon as the compassman has recorded the bearing of the line, he should take the compass and walk rapidly to the next station, marked either by the flag or target, and, if in full view of the chainmen, remove the station mark and set up the compass and be prepared to take the next bearing the moment it is indicated by the chief of party. As soon as the chainmen reach the compass and have "taken the plus" from the last full station, the hind chainman calls out the full station and plus, which the head chainman marks on a fresh stake and which the compassman records as the length of the course run. If the same line is to be continued or "produced," the compass is set at the same bearing as the course just run and the chainmen are lined in by the compassman.

1220. Example of the Use of the Compass in Railroad Work.—Suppose *C A D* in Fig. 257 to be a railroad in operation, and that it has been decided to run a compass line from the point *A* along the valley of the stream *X Y* to the point *B*. The bearing of the tangent *A D* can not be determined by setting up the compass at *A*, on account of the attraction of the rails. The *direction* of this tangent, however, can be obtained by setting up the compass at *A* and sighting to the flag held at *D*. The point *A*, which is the starting point of the line to be run, is marked *0*. Producing the line *A D* 440 feet, the point *E* is reached, which has been previously indicated by the chief of party as a proper place for changing the direction of the line. The compass being set up at *E*, the bearing of the line *A E*, which is the line *A D* produced, is found by sighting to *A*, or, what is preferable, to the point *D*, if that point can be seen. The number of Station *E*, viz., 4 + 40, and the bearing

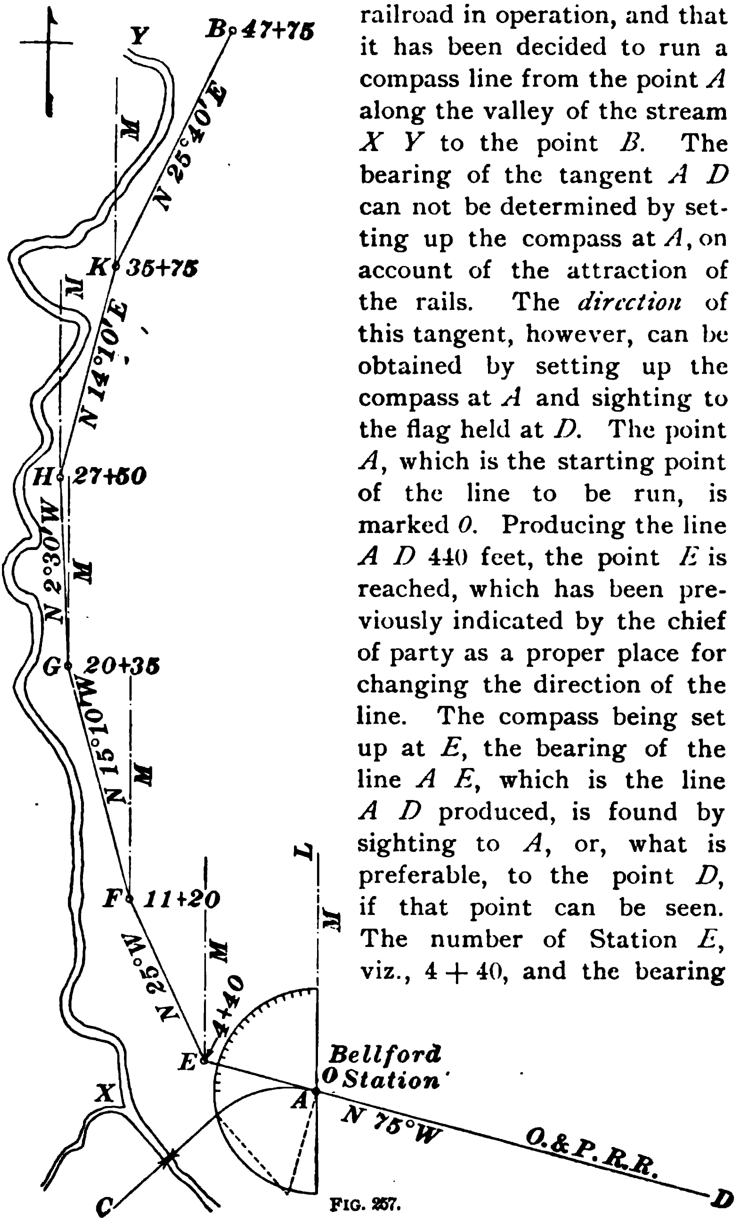


FIG. 257.

of *A E* are then recorded by the compassman. By this time the chief of party has located the point *F*, and the flag is in place for sighting. The axmen, if there is work for them to do, are put in line by the head chainman, clearing only so much as would interfere with rapid chaining. The bearing of the line *E F* being recorded, the compass is moved quickly to *F*, replacing the target left by the flagman, leveled up, and directed toward the point *G*, which is either already, or soon will be, located. The chainmen reaching *F*, its number 11 + 20 is recorded by the compassman, and the instrument sighted to *G* and the work continued as before.

1221. Form for Keeping Notes.—A plain and convenient form for compass notes is the following, which is a record of the survey platted in Fig 257: The first column contains the station numbers, the notation running from the bottom to the top of the page. By such an arrangement, the lengths of the courses are found by subtracting the number of the station of one compass point from the number of the station of the next succeeding compass point. Before commencing the plat, the subtractions are made and the lengths of the courses written in red ink between the station numbers.

The second column contains the bearings of the lines. The bearing recorded opposite to a station is the bearing of the course between the given station and the one next above. Thus, the bearing recorded opposite Sta. 0 is N 75° 00′ W, and is the bearing of the line extending from Sta. 0 to Sta. 4 + 40 next above. The length of the course is the difference between 0 and 4 + 40 equal to 440 ft. The bearing recorded opposite to 4 + 40 is N 25° 00′ W. It is the bearing of the line extending from Sta. 4 + 40 to Sta. 11 + 20 next above. Its length is found by subtracting 4 + 40 from 11 + 20 equal to 680 ft., and so on.

In the third column, under the head of remarks, are recorded notes of reference, topography, and any information which may aid in platting or subsequent location.

Station.	Bearing.	Remarks.
47 + 75		End of line
35 + 75	N 25° 40′ E	
27 + 50	N 14° 10′ E	
20 + 35	N 2° 30′ W	Woodland
11 + 20	N 15° 10′ W	
4 + 40	N 25° 00′ W	
0	N 75° 00′ W	Sta. 0 is at P. C. of 14° curve to left at Bellford Sta. O. & P. R. R.

1222. Platting.—After a survey has been finished, a drawing is made showing the courses. This drawing is called a **plat,** and the operation of making the plat from the field notes is called **platting.**

Since the direction of every line of a compass survey is referred to the same parallel, viz., the magnetic meridian, the readiest mode of platting such a survey is by the use of the **T** square and protractor. The lines drawn to a **T** square are parallel, and in platting take the direction of the magnetic needle, or meridian.

The line *A B*, described in Art. **1220** and shown in Fig. 257, is platted as follows: The arrow shown in the figure gives the direction of the magnetic meridian. A line *A L*, parallel to this meridian, is drawn through the starting point *A*, and from *A* as a center the line *A E*, whose direction N 75° 00′ W is taken from the field notes kept by the compassman, is laid off with a protractor. The directions west are laid off to the left of the meridian, and those east to the right of the meridian. The course *A E*, being a northwest course, is laid off to the left of the meridian *A L*, as shown in the figure. The length of the line *A E* is then measured on this line to any convenient scale, usually 200 feet to the inch, and a parallel to the magnetic meridian drawn through *E*, from which the bearing of the line *E F*, viz., N 25° 00′ W, is laid off and platted. The remaining courses are platted in the same manner.

TRANSIT SURVEYING.

THE INSTRUMENT.

1223. The **engineer's transit,** see Fig. 258, is an instrument in which the telescope takes the place of the plain sights of the compass, and in which the angles are read to single minutes by the vernier. A level *C* is attached to the underside of the telescope and a vertical arc *D* is attached to the outside of the left hand standard. A vernier *E* for reading vertical angles is attached to the telescope axis and adjusted by the tangent screw *F*. The standards *G* and *G*, which support the telescope, are fastened to the upper or vernier plate, as is one of the levels *H*, the other being carried by one of the standards at *H′*. The compass circle *K*, which is divided like that of the ordinary compass, is also a part of the upper plate. The vernier plate covers the lower or divided limb, of which only two small arcs can be seen through the openings where the verniers are placed. A

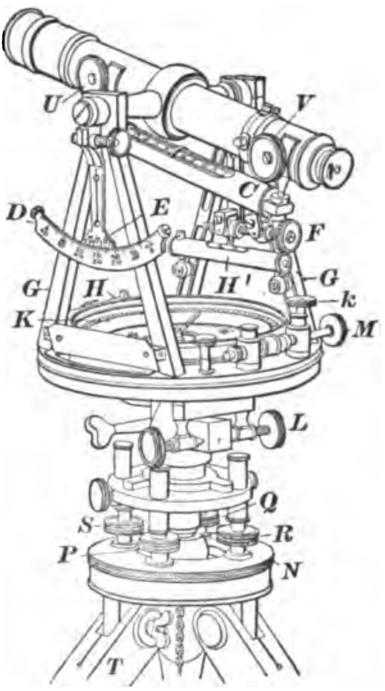


FIG. 258.

screw which clamps the vernier plate to the divided limb is shown at *k*. Slow motion is given to the upper plate by the tangent screw *M*, and to the divided limb by the screw *L*. The transit is fastened to the plate *N* by a ball and socket joint, and is leveled by means of the screws *P*, *Q*, *R*, and *S*. It is fastened to the tripod *T* in a variety of ways, usually screwed to the tripod, the edge of the plate *N* being milled to aid the operator. The transit is brought to center over a point by means of a plumb bob which is suspended by a loop fastened to the lower part of the transit.

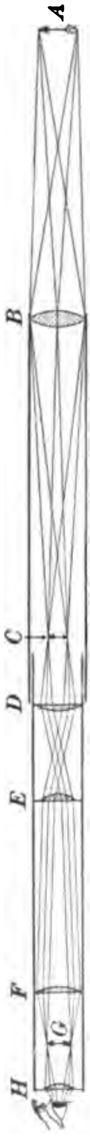


FIG. 259.

1224. The Telescope.—The **telescope** is a combination of lenses placed in a tube and so arranged according to the laws of optics that the image of any object toward which the telescope is directed shall be formed within the tube by the rays of light coming from the object and bent in passing through the object glass. This image is magnified by the eye-piece, which is composed of several lenses. Telescopes are of various kinds, some representing the object erect, i. e., in its natural position, others representing the object inverted.

The telescope shown in Fig. 259 represents the object in an erect position. Rays of light from the object *A* fall upon the object glass *B* where they are bent, and, crossing each other, form the image at *C* in an inverted position. Passing on through the lens *D*, they are refracted or bent, crossing each other again before reaching the lens at *E*. Passing through the lens *F* they form an erect image at *G*, which is in turn magnified by the eye-piece *H*.

1225. The Cross-Hairs.—In order that the line of sight may be precisely brought to bear upon any point of an object within the field of the telescope, two fine lines called **cross-hairs,** or **cross-wires,** are placed with their intersection at the common focus of the object glass and the eye-piece. The intersection of these cross-hairs can be seen through the eye-piece, and seems to be in the same position as that of the image of the distant object.

The line passing through the intersection of the cross-hairs and the optical center of the object glass is called the **line of collimation.**

The cross-hairs are fastened to a thick brass ring placed within the telescope and held in position by **capstan headed screws,** Fig. 260, let into this ring. They are commonly placed at right angles to each other, the one being vertical and the other horizontal. The ring, together with the cross-hairs, can be moved by the capstan headed screws. The cross-hairs are either of platinum wire, drawn very fine, or spider threads. Platinum wire is best, as it is not affected by changes of temperature.

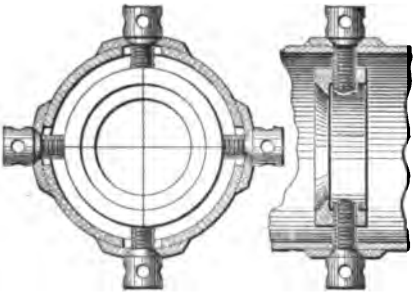


FIG. 260.

1226. Focusing the Telescope.—The movement of the object glass is effected by a milled headed screw *U*, shown in Fig. 258. This screw moves the object glass out or in, according as the object is nearer or further from the instrument. The eye-piece is focused upon the cross-hairs by a similar screw *V*. The cross-hairs are not in proper focus until they appear to be a part of the object looked at, showing no movement, however the position of the eye may be changed.

The telescope is supported upon an axis and so placed that both ends shall be as nearly balanced as possible. The axis rests on upright legs called the **standards.** The standards are fastened to the upper plate.

1227. The Graduated Circle.—This circle is divided into 360 equal parts or degrees, and each degree is further divided into two or three equal parts. If the degree is divided into two equal parts, each part equals 30′, and if into three equal parts, each part equals 20′. The degrees number from 0 to 360°, and in most instruments there is an inner graduated circle, which numbers each way from 0 to 90°, as on the compass circle. Each tenth degree is numbered; each fifth degree is indicated by a longer line of division, and each degree by a line longer than its subdivisions.

1228. Movements.—When the line of sight is to be brought to bear upon a distant object, the observer turns the telescope in the direction of the object by lightly but firmly grasping the upper plate, one hand on either side of the instrument. The eye is ranged along the top of the telescope, which is turned by the hands until it appears to be in the direct line of the object. The eye is then brought to the eye-piece, and the object glass focused upon the object. The instrument is then clamped, and, by means of either of the tangent screws, the cross-hairs are brought to bear precisely upon any desired point of the object viewed.

1229. The Levels.—Most of the angles measured by the transit are horizontal angles, but whether horizontal or vertical, before an angle can be measured, the plate carrying the graduated circle must be brought to a horizontal position. This is effected by means of two small levels placed on the plate at right angles to each other. Each level consists of a glass tube curved upwards at its middle and nearly filled with alcohol, leaving only space for a bubble of air. They are so placed that when the air bubbles are exactly in the middle of the tubes, the plate upon which they rest will be in a level position. The leveling is performed by means of four leveling screws. They have milled heads and are arranged in pairs, the line passing through one pair being at right angles to that passing through the other pair.

1230. To Level the Instrument.—Loosen the lower clamp and bring one of the bubble tubes into a parallel to a plane passing through a pair of opposite screws. By turning these screws, the air bubble can be brought exactly to the center of the tube. As the tubes are at right angles to each other, the putting of one in position for leveling adjusts the other for leveling also, and having leveled one tube with one pair of screws, the other tube is leveled with the other pair.

1231. The Vernier.—A **vernier** is a contrivance for measuring smaller portions of space than those into which

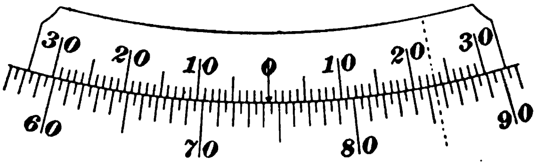


FIG. 261.

a line is actually divided. The divided circle of the transit is graduated to half degrees, or 30′. The graduations on the verniers run in both directions from its zero mark, making two distinct verniers, one for reading angles turned to the right, and the other for reading those turned to the left. Each vernier is divided into 30 equal spaces, which are together equivalent to 29 spaces on the divided circle; hence, each space on the vernier is equal to 29′, and the vernier is described as reading to minutes. In reading the vernier, the observer should first note in which direction the graduations of the divided circle run. In Fig. 261 the graduations increase from left to right and extend from 57° to 91°. Next he should note the point where the zero mark of the vernier comes on the divided circle. In Fig. 261, the zero mark comes between 74° and 74½°. Now, as the circle graduations read from left to right, we read the right-hand vernier and find that the 23d graduation on the vernier coincides with a graduation on the divided circle, and the vernier reads 23′, which we add to 74°, making a reading of 74° 23′, an angle to the *left*. In Fig. 262 the graduations on the circle increase from right to left, and we accordingly read the left-hand vernier. The zero mark of the vernier comes between 67½ and 68°. Reading the vernier, we find that the 13th graduation on the vernier coincides with a graduation on the circle, and the vernier reads 13′. Accordingly, we add to 67½°, the vernier reading of 13′, making a total reading of 67° 43′, an angle to the *right*.

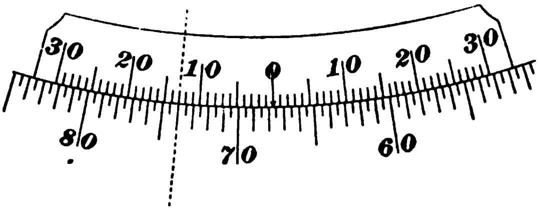


FIG. 262.

ADJUSTING THE TRANSIT.

1232. The constant use of an instrument tends to disarrange some of its parts, which detracts from the accuracy of its work, without in any way injuring the instrument itself.

The correction of this disarrangement of parts is called making the adjustments.

The transit, when leveled up, will, if in adjustment, fulfil the following conditions, viz.:

1. It will maintain a perfectly **horizontal position** during an **entire revolution.**

2. The line of sight, when directed in **opposite directions,** will be in the **same straight line ;** and

3. The line of sight will revolve in a **vertical plane** perpendicular to the **horizontal plane** of revolution.

The adjustments should be made in the order of these three conditions. The best time of the day for making the adjustments, especially in the summer season, is the early morning, before the air has become heated and the sun dazzling.

1233. First Adjustment.—Secure, if possible, an open space where a clear sight may be had for at least 400 feet in both directions from the transit. Plant the feet of the tripod firmly in the ground, and then bring the plate to a horizontal position with the leveling screws. Next turn the vernier plate half way around, i. e., revolve it through an angle of 180°. If the bubbles are in adjustment they will remain stationary in the centers of the tubes. If they do not remain so, but run to either end, bring them half way back to the middle of the tubes by means of the capstan headed screws attached to the tubes, and the rest of the way back by the leveling screws. Then, revolve them again through 180°. Sometimes this adjustment is made by one trial, but it is usually necessary to repeat the operation.

1234. Second Adjustment.—To cause the line of collimation to revolve in a plane:

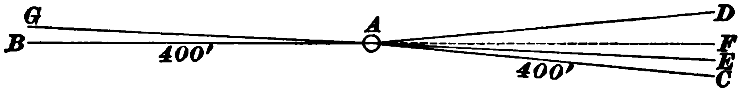


FIG. 263.

Measure from A, where the instrument is stationed (see Fig. 263), 400 feet to the point B, where a pin (or tack, if it can be seen) is fixed.

Carefully direct the line of sight to this point, and **reverse** the telescope, i. e., *turn it on its axis until it points in the opposite direction.* If the line of collimation is "in adjustment," a pin set 400 feet from A, on the opposite side of the instrument from B, will be at F and in the same line as A B; if it is not in adjustment, the pin will be on one side of F, as at D. Turn the *vernier plate* half way around, that is, through 180°, and direct the line of sight again to B.

Reverse the telescope, and the pin will be at *C*. Carefully measure the distance *C D*, and at *E*, one-fourth of the distance from *C* to *D*, set the pin. Move the cross-hairs by means of the capstan headed screws until the vertical hair shall exactly cover the pin at *E*, being careful to move them in the opposite direction from that in which it would appear they should move. This movement having been made and the telescope reversed, the line of sight will not be at the point *B*, but at *G*, a distance from *B* equal to *C E*. Again sight to *B*, and, reversing, the pin will be at *F*, in the same line as *A B*. It may be necessary to repeat the operation to secure an exact adjustment. If so, take a new set of points, a few inches removed from those first used, to avoid confusion.

1235. Third Adjustment.—To cause the line of collimation to revolve in a vertical plane:

Suspend a plumb bob at as high an elevation as can be readily found; direct the line of sight to the upper end of this line and then, revolving the telescope slowly downwards, see if the intersection of the cross-hairs closely follows this line throughout its length. If it does follow it, the line of collimation revolves in a vertical plane. If it does not, the adjustment may be made as follows: Take a point *A*, in Fig. 264, on a church spire or some other high object, and sight carefully to it. Depress the telescope until a pin can be set in the ground at its base, as at *B*. Loosen the clamp and turn the plate through 180° without touching the telescope. Clamp the instrument and sight again to the high point *A*. Again depress the telescope and set another pin, which it will be found is at some distance from *B*, as at *C*. The vertical plane is the line *A D*, and it will be seen that the error is doubled. The adjustment is made by raising or lowering one end of the telescope axis by means of a small screw placed in the standard for that purpose.

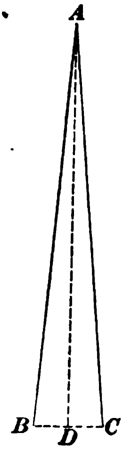


FIG. 264.

DIRECTIONS FOR USING THE TRANSIT.

1236. Care of the Transit.—The transit, though it will bear a lifetime of legitimate service, will not stand neglect or banging. The bearings are delicate and easily marred by particles of dust or sudden blows. Moisture clouds the lenses, and, when combined with dust, is doubly injurious. Little advantage is gained from working in the rain, and, unless the stress of work requires it, both instrument and men are better off under cover. If the instruments should encounter a wetting, carefully wipe the object glass, eye-piece, and verniers with a piece of chamois skin, as moisture soon clouds them so as to prevent further work. As soon as the party returns to office or camp, complete the drying process by thoroughly rubbing with a piece of chamois skin, which every engineering party should carry. When a party rides to and from work, the instruments should be carried in their cases, and they should always be kept in their cases when in the office. The common custom of leaving an instrument on its tripod and standing on a board floor can not be too severely condemned.

1237. Setting Up the Instrument.—As much of the work of an engineering party is suspended while the instrument is being set up, it is highly important to acquire facility in setting it up. The following suggestions will be of use, although practice alone will make one expert.

In setting up a transit, three preliminary conditions should be met as nearly as possible, viz.:

1. The tripod feet should be firmly planted.
2. The plate on which the leveling screws rest should be level; and
3. The plumb bob should be directly over the given point.

The third condition must be met to a nicety, and this is rendered comparatively easy by means of a "shifting head" with which most modern transits are provided. When these three conditions are approximately met, the completion of the operation is quickly performed with the leveling screws.

1238. How to Prolong a Straight Line.—Let *A B*, in Fig. 265, be a straight line, and it is required to prolong or produce it 400 feet to *C*.

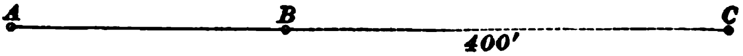


FIG. 265.

The line can be prolonged in two ways—by means of **foresight** and **backsight.**

1. By foresight, set up the transit at *A* and sight to *B*; let the chainman measure 400 feet from *B* in the direction in which the line is to be prolonged. Then, by means of signals, move the flag to right or left until the vertical cross-hair shall exactly divide the flag held at *C*. Then, the line *B C* will be the prolongation of the line *A B*.

2. By backsight, set the transit at *B* and sight to *A*. Reverse the telescope, and, having measured 400 feet from *B* in the opposite direction from *A*, set the flag at *C*, then the line *B C* will be the line *A B* produced.

1239. Double Centers.—In prolonging lines, a device known as **double centering** is sometimes used. It is unnecessary when using an instrument that is in proper adjustment, but it is a good check, and a knowledge of the method is valuable.

Let *A B*, in Fig. 266, be a given line which it is required to produce 1,000 feet. Set up the transit at *B*;

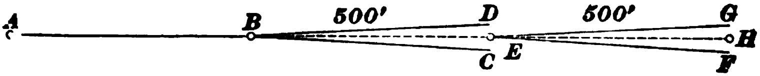


FIG. 266.

backsight to *A*, and reverse the instrument. Set a point *C* 500 feet from *B*. Unclamp the upper plate and revolve the telescope through 180°, backsighting again to *A*. Reverse the telescope. If the line of sight does not come at *C*, then the point *C* is not in line with the points *A* and *B*, and the line of sight will be at some point, as *D*, on the opposite side of the true line. Measure the space *C D* and mark its middle point *E*. The point *E* will be in the prolongation of the line *A B*. Move the transit to *E*, and, backsighting to *B*, determine the point *H* by the same means used in fixing the point *E*.

1240. Horizontal Angles and Their Measurement.—A horizontal angle is one the boundary lines of which lie in the same horizontal plane. Let *A*, *B*, and *C*, in Fig. 267, be three points, and let it be required to find the horizontal angle formed by the lines *A B* and *A C* joining these points. Set up the instruments precisely over the angular point *A*, and carefully level it. Set the vernier at zero, and place the flag at *B* and at *C*. Sight the flag at *B* and set the lower clamp. Then, by means of the lower tangent screw cause the vertical cross-hair to exactly bisect the flag at *B*. Loosen the upper clamp. With a hand on either standard, turn the telescope in the same direction as that of the hands of a watch until the flag at *C* is covered or nearly covered by the vertical cross-hair. Clamp the upper plate and with the upper tangent screw bring the line of sight exactly on the flag at *C*. The arc of the graduated circle traversed by the zero point of the vernier will be the measure of the angle *B A C*, equal to 143° 30′. The points *A*, *B*, and *C* are not necessarily in the same horizontal plane, but the level plate of the instrument projects them into the horizontal plane in which it revolves.

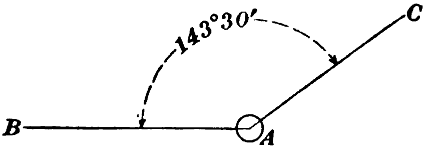


FIG. 267.

1241. A Deflected Line.—A deflected line, or "angle line," is a consecutive series of lines and angles. The direction of each line is referred to the line immediately preceding it, which preceding line is, in imagination, produced, and the angle measured between it and the next line actually run. The angles are recorded *R′* or *L′*, according as they are turned to the right or left of the prolongation of the immediately preceding line. An example of a deflected line

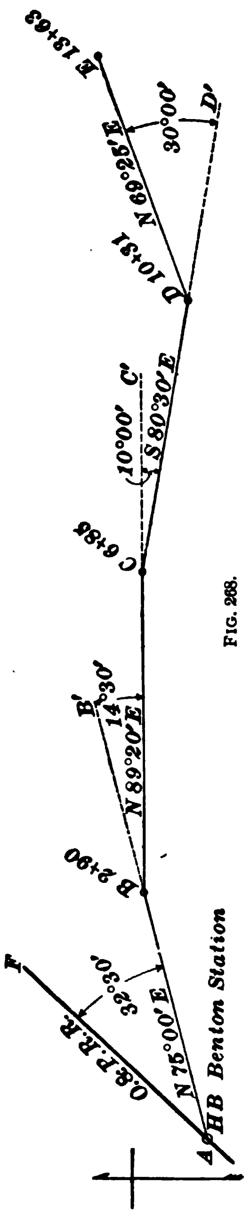
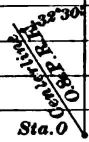


FIG. 268.

is shown in Fig. 268. Here the starting point, *A*, of the line is a point in the *head-block* of the switch at Benton Station, O. & P. R. R. The point *A* is, of course, in the center line of the track.

Set up the transit at *A* with the vernier at zero. Sight to a flag held at *F* on the center line of the track, O. & P. R. R. Loosen the vernier clamp, and turn the telescope sight to a flag held at *B*, the next point on the angle line; clamp the vernier, and, by means of the tangent screw, accurately sight to the flag held at *B*; the angle reads 32° 30', and is recorded *R'* 32° 30', with a sketch showing the connection in which the term head-block is designated by the abbreviation *H. B.* The bearing of the line *A B* can not be taken at *A* on account of the attraction of the rails. The instrument is now moved to *B*, the vernier set at zero and backsighted to *A*; the bearing of *A B*, N 75° 00' E, is taken, and the number of station *B*, 2 + 90, together with the bearing of *A B*, recorded. The telescope is then reversed, pointing in the direction *B B'*. The point *C* being determined, the upper clamp is loosened and the telescope turned to the right and sighted to *C*. The reading of the angle is found to be 14° 30', and recorded *R'* 14° 30'. It measures the angle *B' B C*. The bearing of the line *B C*, N 89° 20' E, is then recorded. The instrument is next set up at C, the vernier set at zero, backsighted to B, and then reversed; the deflection to D, R^t 10° 00′, is then read and recorded, together with the number of the station at C, 6 + 85. This deflection measures the angle $C' C D$, and

Station.	*Deflection.*	*Mag. Bearing.*	*Ded. Bearing.*	*Remarks.*
13+63				*End of Line.*
10+31	*L^r 30°00′*	*N. 69°25′ E.*	*N. 69°30′ E.*	
6+85	*R^r 10°00′*	*S. 80°30′ E.*	*S. 80°30′ E.*	
2+90	*R^r 14°30′*	*N. 89°20′ E.*	*N. 89°30′ E.*	*H.B. of Switch*
0		*N. 75°00′ E.*		*at Benton Sta.*

gives the direction of the line $C D$. A good form of notes for such a survey is that given above.

1242. Checking Angles by the Needle.—In spite of the greatest care, errors in the reading and recording of angles will occur. The best check to such errors is the magnetic needle. And though it is not an exact check, owing to the lack of precision in reading the needle and to local attraction, yet it is the only reliable one, and in universal use.

In Fig. 269, we have an example of the use of the needle

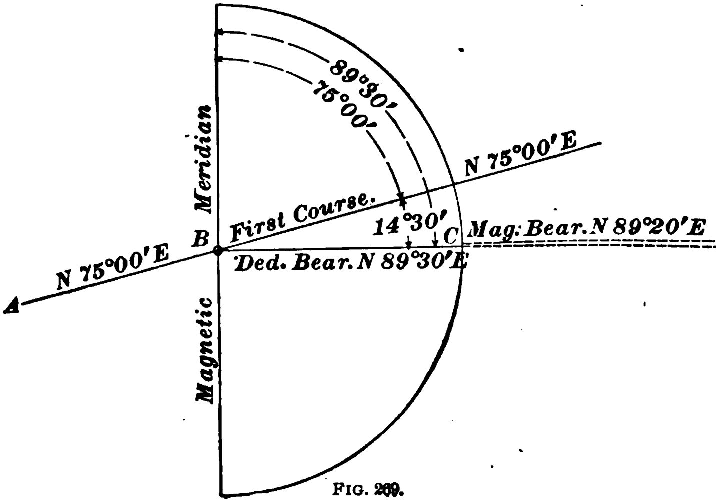


FIG. 269.

in checking angles. The bearing of the line $A\ B$, which corresponds to $A\ B$ in Fig. 268, is N 75° 00′ E, and is assumed to be correct. The bearing of the line $B\ C$, as read from the needle, is N. 89° 20′ E. Its **deduced** or **calculated bearing** is obtained as follows: To the bearing of the line $A\ B$, N 75° 00′ E, we add the R' deflection 14° 30′; the sum is 89° 30′, which is recorded in the column headed Ded. Bearing. (See Art. **1241.**) The deduced bearing, it will be seen, is ten minutes greater than the magnetic bearing as read from the needle and recorded in the column headed Mag. Bearing. Had the deflection angle been recorded L' instead of R', the deduced bearing would have been the difference between 75° 00′ and 14° 30′, which is 60° 30′, and would be recorded N 60° 30′ E. The magnetic bearing being N 89° 20′ E would have at once revealed the error. The confusion of the directions R' and L' is the commonest source of error in recording deflections, though sometimes a mistake of ten degrees is made in reading the vernier. It is a wise precaution to read both angle and bearing after they are recorded and compare them with the recorded readings.

TRIANGULATION.

1243. Simple Triangulation.—Triangulation is an application of the principles of trigonometry to the measurement of inaccessible lines and angles. A common occasion for the use of trigonometry is illustrated in Fig. 270, where the line of survey crosses a stream too wide and deep for actual measurement. Set two points A and B on line, one on

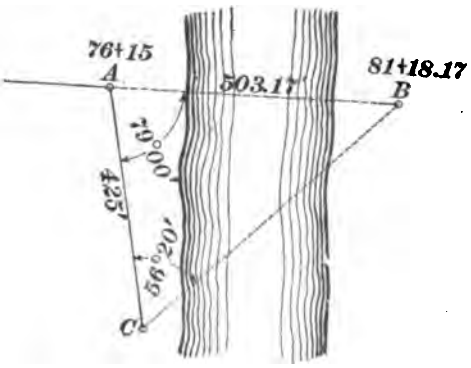


FIG. 270.

each side of the stream. Estimate roughly the distance *A B*. Suppose the estimate is 425 feet. Set another point *C*, making the distance *A C* equal to the estimated distance *A B* = 425 feet. Set the transit at *A* and measure the angle *B A C* = say 79° 00′. Next set up at the point *C* and measure the angle *A C B* = say 56° 20′. The angle *A B C* is then determined by subtracting the sum of the angles *A* and *C* from 180°; thus, 79° 00′ + 56° 20′ = 135° 20′. 180° 00′ − 135° 20′ = 44° 40′ = the angle *A B C*. We now have a side and three angles of a triangle given, to find the other two sides *A B* and *C B*. These sides may be easily found by the methods given for the solution of triangles (see Arts. **759**, etc.) by drawing a line from the vertex of one of the angles *A* or *C* so as to divide the triangle *A B C* into two right-angled triangles. A simpler and easier method, however, is the following: In higher works on trigonometry, it has been demonstrated that, *in any triangle, the sines of the angles are proportional to the lengths of the sides opposite to them.* In other words, sin *A* : sin *B* :: *B C* : *A C*; or, sin *A* : sin *C* :: *B C* : *A B*, and sin *B* : sin *C* :: *A C* : *A B*.

Hence, we have sin 44° 40′ : sin 56° 20′ :: 425 : side *A B*.

Sin 56° 20′ = .83228;
.83228 × 425 = 353.719;
sin 44° 40′ = .70298;
353.719 ÷ .70298 = 503.17 ft. = side *A B*.

Adding this distance to 76 + 15, the station of the point *A*, we have 81 + 18.17, the station at *B*.

Another and frequent occasion for the use of trigonometry is the following: Two tangents, *A B* and *C D*, Fig. 271, which are to be united by a curve, meet at some inaccessible point *E*. Tangents (which will be more fully described later on) are the straight portions of a line of railroad. The angle *C E F*, which the tangents make with each other, and the distances *B E* and *C E* are required. Two points *A* and *B* of the tangent *A B*, and two points *C* and *D* of the tangent *C D*, being carefully located, set the transit at *B*, and, backsighting to A, measure the angle $E\,B\,C = 21^\circ\ 45'$; set up at C, and backsighting to D, measure the angle $E\,C\,B = 21^\circ\ 25'$. Measure the side $B\,C = 304.2$ ft.

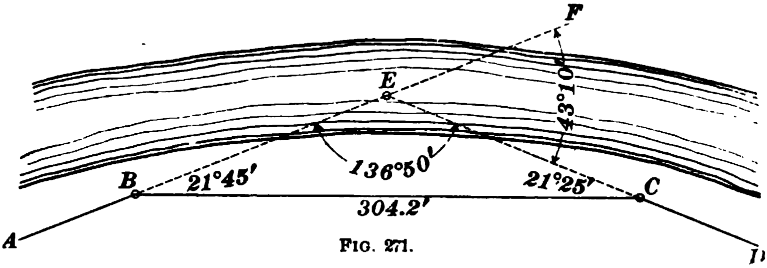


FIG. 271.

Angle $C\,E\,F$ being exterior to the triangle $E\,B\,C$ is equal (see Art. **1194**). to the sum of $E\,B\,C$ and $E\,C\,B = 21^\circ\ 45' + 21^\circ\ 25' = 43^\circ\ 10'$. The angle $B\,E\,C = 180^\circ - C\,E\,F = 136^\circ\ 50'$.

From the principle stated we have sin 136° 50′ : sin 21° 45′ :: 304.2 ft. : side $C\,E$.

$\sin 21^\circ\ 45' = .37056$;
$.37056 \times 304.2 = 112.724352$;
$\sin 136^\circ\ 50' = .68412$;
side $C\,E = 112.724352 \div .68412 = 164.77$ ft.

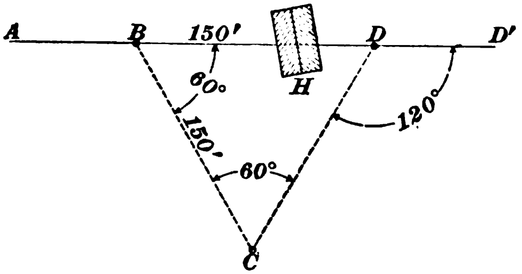


FIG. 272.

Again, we find $B\,E$ by the following proportion:

Sin 136° 50′ : sin 21° 25′ :: 304.2 : side $B\,E$;
$\sin 21^\circ\ 25' = .36515$;
$.36515 \times 304.2 = 111.07863$;
$\sin 136^\circ\ 50' = .68412$;
side $B\,E = 111.07863 \div .68412 = 162.36$ ft.

A building *H*, Fig. 272, lies directly in the path of the line *A B* which must be produced beyond *H*. Set a plug at *B* and then turn an angle *D B C* = 60°. Set a plug at *C* in the line *B C*, at a suitable distance from *B*, say 150 feet. Set up at *C*, and turn an angle *B C D* = 60°, and set a plug at *D*, 150 ft. from *C*. The point *D* will be in the prolongation of *A B*. Then, set up at *D* and backsighting to *C*, turn the angle *C D D'* = 120°. *D D'* will be the line required, and the distance *B D* will be 150 feet, since *B C D* is an equilateral triangle.

A B and *C D*, Fig. 273, are tangents intersecting at some

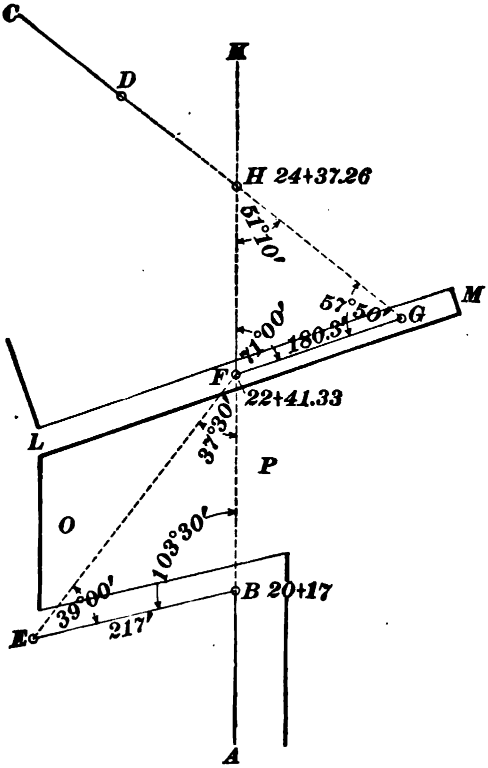


FIG. 273.

inaccessible point *H*. The line *A B* crosses a dock *O P*, too wide for direct measurement, and the wharf $L M$. F is a point on the line $A B$ at the wharf crossing. It is required to find the distance $B H$ and the angle $F H G$. At B, an angle of 103° 30′ is turned to the left and the point E set 217′ from B = to the estimated distance $B F$. Setting up at E, the angle $B E F$ is found to be 39° 00′. Whence, we find the angle $B F E = 180° - (103° 30' + 39°) = 37° 30'$. From the above principle we have sin 37° 30′ : sin 39° 00′ :: 217 ft. : side $B F$.

$$\begin{aligned}&\text{Sin } 39° 00' = .62932;\\&.62932 \times 217 = 136.56244;\\&\text{sin } 37° 30' = .60876;\\&\text{side } B F = 136.56244 \div .60876 = 224.33 \text{ ft.}\end{aligned}$$

Whence, we find the station of F to be 20 + 17 + 224.33 = 22 + 41.33. Set up at F and turn an angle $H F G = 71° 00'$, and set up at a point G where the line $C D$ prolonged intersects $F G$. Measure the angle $F G H = 57° 50'$, and the side $F G = 180.3'$. The angle $F H G = 180° - (71° + 57° 50') = 51° 10'$. From the same principle as before we have sin 51° 10′ : sin 57° 50′ :: 180.3′ : side $F H$.

$$\begin{aligned}&\text{Sin } 57° 50' = .84650;\\&.84650 \times 180.3 = 152.62395;\\&\text{sin } 51° 10' = .77897;\\&\text{side } F H = 152.62395 \div .77897 = 195.93 \text{ ft.};\end{aligned}$$

whence, we find the station of H to be 24 + 37.26.

1244. Vertical Angles.—A **vertical angle** is an angle formed by two intersecting lines lying in the same vertical plane, one of which is horizontal. If the lines $A B$ and $A C$, Fig. 274, lying in the vertical plane $D E F G$, meet at the point A, and the line $A B$ is horizontal, the angle $C A B$ is a *vertical angle*, and is measured by the arc $B C$.

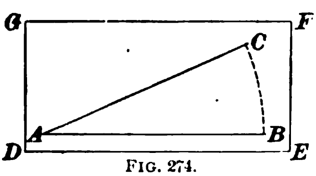


FIG. 274.

1245. Intersection of Tangents.—Let $A B$ and $C D$, Fig. 275, be tangents whose point of intersection is to be determined and the angle which they make with each other to be measured. First set up a flag or stake at *B* and another at *A*, or some other point in the line *A B*. Set up the transit at *C*, backsighting to *D*. Reverse the instrument. Have a flagman hold a rod in the line *C D*, at the same time putting himself in range with the stakes at *A* and *B*. With a little practice he can nearly determine the intersection *I* of the two lines. Then drive two stakes *K* and *L* firmly in the line *C D*, one on each side of the point *I*. Their distance from the point *I* to be determined by the obtuseness of the angle *A I D*. Carefully center these stakes, driving a tack half its length in each center. Stretch a cord between these tacks. Next set up the instrument at *B*, backsighting to *A*. Reversing the telescope, set a flag at *I*, which will be the intersection of the line *A B* prolonged with *L D*. Drive a stake flush with the ground at *I* and drive a tack in this stake where the prolongation of *A B* crosses the cord connecting the stakes at *K* and *L*. The point *I* is the intersection of the tangents *A B* and *C D*. The external angle *C I M*, formed by the intersecting tangents, is called the **angle of intersection.**

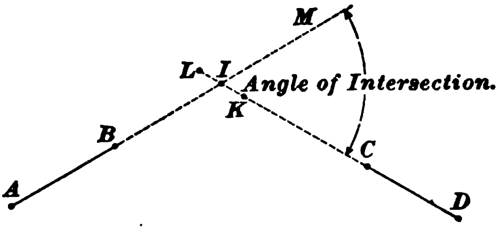


FIG. 275.

CURVES.

1246. A line of railroad consists of a series of straight lines and curves. In general, the straight lines, or, more properly, the tangents, are first located and then they are united by curves best fitting the ground lying between the tangents. There are certain limits of curvature prescribed for all roads, which must not be exceeded. These limits will depend upon conditions to be explained later. Railroad curves are circular and are divided into *simple*, *compound*, and *reverse* curves.

A **simple curve** has but one radius, as *A B* in Fig. 276, whose radius is *A C*.

A **compound curve,** shown in Fig. 277, is a continuous curve of two or more arcs of different radii, as *C D E F*, which is composed of the arcs *C D*, *D E*, and *E F*, whose respective radii are *G C*, *H D*, and *K E*.

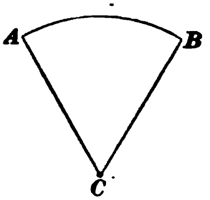


FIG. 276.

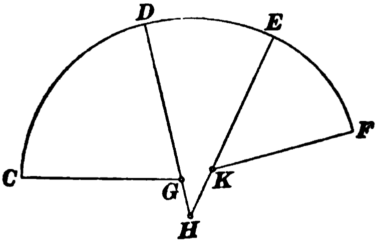


FIG. 277.

A **reverse curve,** Fig. 278, is a continuous curve composed of two arcs *L M* and *M N* of the same or of different radii described in the opposite directions, and having a common point *M*, called the point of reversal. Reverse curves, though common in the early days of railroad building in the United States, are now condemned for roads of standard gauge, and only admitted for narrow-gauge roads, when cheapness of construction is the first requirement.

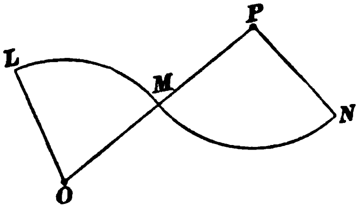


FIG. 278.

1247. Geometry of the Circle.—Before attempting to lay out curves, a knowledge of geometry relating to the circle must be mastered. The following propositions are of special importance:

1. A tangent to a circle is perpendicular to the radius drawn through its tangent point. Thus, *A E*, Fig. 279, is perpendicular to *B O*, and *C E* is perpendicular to *C O*.

2. Two tangents drawn to a circle from any point are equal, and if a chord be drawn joining these points, the angles between the chord and the tangents are equal. Thus, *B E* and *C E* are equal, and the angles *E B C* and *E C B* are equal.

3. An acute angle between a tangent and a chord is equal to half the central angle subtended by the same chord; thus, the angle $E B C = E C B =$ one-half *B O C*.

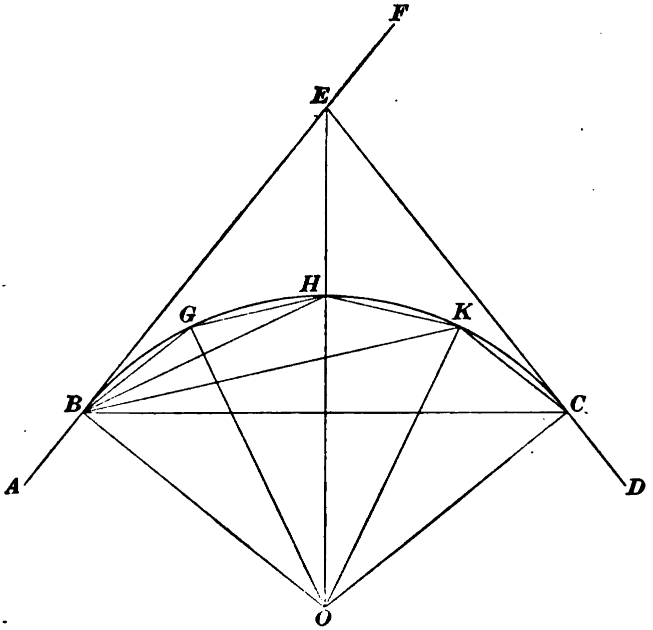


FIG. 279.

4. An acute angle subtended by a chord, having its vertex in the circumference of a circle, is equal to half the central angle subtended by the same chord. Thus, the angles *E B G*, *G B H*, etc., having the vertex *B* on the arc *B D* and subtended by the chords *B G*, *G H*, etc., are equal to half the central angles *B O G*, *G O H*, etc., subtended by the same chords.

5. Equal chords subtend equal angles at the center of a circle and also at the circumference, if the angles are

inscribed in similar segments. Thus, if *B G*, *G H*, *H K*, and *K C* are equal, *B O G* = *G O H* and *G B H* = *H B K*.

6. The angle of intersection of two tangents equals the central angle subtended by the chord uniting the tangent points. Thus, the angle *C E F* = *B O C*.

1248. Deflection Angles.—When two lines meet in the same plane, they are said to form an angle, and the point of meeting is called the **angular point.** The rate of divergence or deflection of the two lines from their common or angular point determines the size of the angle. The unit of angular measurement is the *degree*, equal to $\frac{1}{360}$ part of a circle. Two lines forming an angle of one degree with each other will, at a distance of one hundred feet from the angular point, deflect or diverge 1.745 feet.

In Fig. 280, the lines *A B* and *A C*, meeting at the point *A*, are supposed to form an angle of 1°, and the angle *B A C* is measured by the arc *B C*, described with the radius *A B*, which is 100 feet in length. The arc *B C* and the straight line joining the extremities of that arc, i. e., the chord *B C*, are assumed to be of equal length.

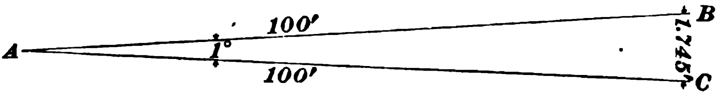


FIG. 280.

1249. Degree of Curvature.—The curve from which, as a unit or basis, all other railroad curves are deduced, is called a **one-degree curve.** It is the circumference of a circle whose radius is 5,730 feet, or, more exactly, 5,729.65 feet, in length. Two radii forming an angle of one degree at the center of a one-degree curve will subtend a chord of 100 feet at its circumference. The arc subtended by this chord of 100 feet is assumed to be of the same length as the chord.

In Fig. 281, let *A B* and *A C* be radii 5,729.65 feet in length, forming an angle of 1° at the center *A*; then the arc *B C* subtended by these radii will be 100 feet in length. The curve *B C* is called a 1° curve. If, from the point *O* as a center, with a radius *O B* equal to 2,864.93 feet, we describe an arc *B D* 100 feet in length, the radii *O B* and *O D* will

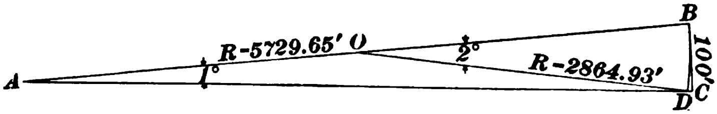


FIG. 281.

form an angle of 2° at the center *O*, and the curve *B D* is called a 2° curve. A curve whose radius is nearly one-third *A B*, or 1,910.08 feet, is a 3° curve, etc.

The **degree of a curve** is determined by the central angle, which is subtended by a chord of 100 feet. Thus, if *B O G* (Fig. 282) is 10° and *B G* is 100 feet, *B G H K C* is a 10° curve.

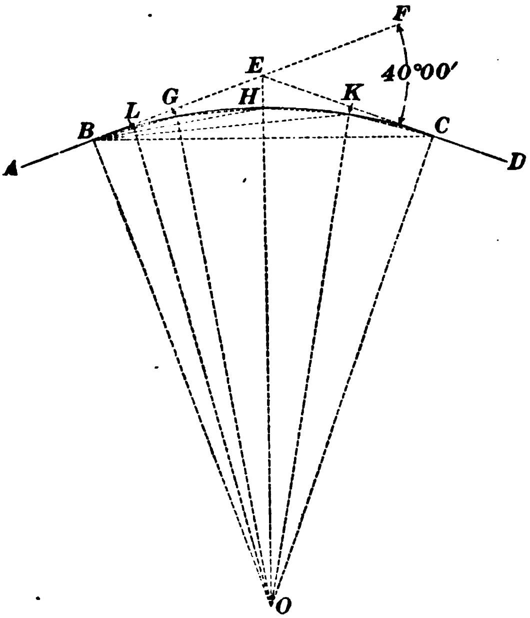


FIG. 282.

The **deflection angle** of a curve is the angle formed at any point of the curve between a tangent and a chord of 100 feet. The deflection angle is, therefore, **half the degree**

of the curve. Thus, if the chord $B\,G$ is 100 feet, the angle $E\,B\,G$ is the deflection angle of the curve $B\,G\,H\,K\,C$, and is half the angle $B\,O\,G$.

Example.—Given the deflection angle $E\,B\,G = D$ (Fig. 282), to find the radius $B\,O = R$.

Solution.—Draw $O\,L$ perpendicular to $B\,G$. In the right-angled triangle $B\,O\,L$, we have $\sin B\,O\,L = \frac{B\,L}{B\,O}$; but $B\,O\,L = E\,B\,G = D$, since $O\,L$ being perpendicular to the chord $B\,G$ it bisects the arc $B\,L\,G$. But the angle $D = \frac{1}{2}\,B\,O\,G$; hence, angle $B\,O\,L = D$. $B\,L =$ 50 feet and the radius $B\,O = R$. Substituting these values in the given equation, we have $\sin D = \frac{50}{R}$; whence, $R \sin D = 50$, and we have the formula

$$R = \frac{50}{\sin D}. \qquad (89.)$$

For curves of from 1° to 10°, the radius may be found by dividing 5,730 ft. (the radius of a 1° curve) by the degree of the curve. The results obtained are sufficiently accurate for all practical purposes. For sharp curves, i. e., for those exceeding 10°, the above formula, viz., $R = \frac{50}{\sin D}$ should be used, especially if the radii are to be used, as a basis for further calculation.

For example, the radius of a 4° curve is found by both methods as follows: By first method, $R = 5{,}730$ ft. $\div\ 4 = 1{,}432.5$ ft. By second method, we find the deflection angle D of a 4° curve is 2°. Applying the formula, $R = \frac{50}{\sin D}$, we have $R = \frac{50}{.0349} = 1{,}432.67$ ft.

In this case the error is only .17 foot, and may be ignored in practical work. For a 30° curve we have by first method, $R = \frac{5{,}730}{30} = 191$ ft. By second method, we have $R = \frac{50}{\sin 15^\circ} = \frac{50}{.25882} = 193.18$ ft. In this case the error is 2.18 ft., and the error increases as the degree of curve increases.

The *radii* given in the table of Radii and Deflections are calculated by the formula $R = \frac{50}{\sin D}$.

1250. Sub-Chords for Curves of Short Radii.—On curves of short radii, i. e., curves of 20° and upwards, center stakes are driven at intervals of 25 feet. In Art. **1248,** we stated that the standard chord and arc are assumed to be of the same length. This is practically true for curves of large radii, but for curves above 20° the excess of length of arc over the chord constantly increases. If, now, in Fig. 282, the chord *B C* is 100 feet in length, the arc *B G H K C* must be *greater* than 100 feet: and if the arcs *B G*, *G H*, *H K*, and *K C* are equal, i. e., each equal to one-quarter the arc *B H C*, then the equal chords *B G*, *G H*, *H K*, and *K C* subtending these equal arcs must each be *greater* than one-quarter of *B C*, which we assumed to be 100 feet. These greater chords must, therefore, be *greater than* 25 feet. Suppose the curve *B H C* to be a 20° curve, and the chord *B C*, 100 feet; then the central angle *B O C* is 20°. As the arc *B G* is one-quarter of the arc *B H C*, the central angle *B O G* is $\frac{20^\circ}{4} = 5^\circ$. The line *O L*, drawn to the middle point of the chord *B G*, is perpendicular to *B G* and bisects the angle *B O G*. The deflection angle $E\,B\,G = B\,O\,L = G\,O\,L$. Let *C* designate the chord *B G*, *R*, the radius *O B* and *D*, the deflection angle, $E\ B\ G = B\ O\ L$. In the right-angled triangle *B O L*, we have $\sin B\ O\ L = \frac{B\ L}{B\ O}$. Substituting the above given values, we have $\sin D = \frac{\frac{1}{2} C}{R}$, whence $\frac{1}{2}\ C = R \sin D$, and we have

$$C = 2\ R \sin D. \qquad \textbf{(90.)}$$

The central angle for the chord *B G* is 5°. The deflection angle *D* is, therefore, $\frac{5}{2}^\circ = 2^\circ\ 30'$. $\sin 2^\circ\ 30' = .04362$. Since the deflection angle $E\ B\ C = 10^\circ$ for this case; $R = 50 \div \sin 10^\circ = 287.94$ ft. Hence, chord $C = 2 \times 287.94 \times .04362 = 25.12$ ft.

Accordingly, in measuring the short chords, 25.12 feet should be used instead of 25 feet.

1251. Tangent Distances.—When an intersection of tangents has been made and the intersection angle measured, the next question is the degree of curve which is to unite them, which being decided, the next step in order is the location of the points on the tangents where the curve begins and ends. These two points are equally distant from the point of intersection of the tangents, which is called the P. I. The point where the curve begins is called the **point of curve**, or the P.C.; the point where the curve terminates is called the **point of tangent**, or the P. T. The distance of the P. C. and P. T. from the P. I. is called the **tangent distance.**

In Fig. 282, let $A\,B$ and $C\,D$ be tangents intersecting at the point E and forming an angle $C\,E\,F = 40°\ 00'$ with each other. It is decided to unite these tangents by a 10° curve whose radius is 573.7 feet. Call the angle of intersection I, the radius $B\,O$, R, and the tangent distance $B\,E$, T. From Art. **1247,** proposition 6, we have $B\,O\,C = C\,E\,F$; hence, the angle $B\,O\,E = \frac{1}{2}\,C\,E\,F$. From the right triangle $E\,B\,O$ we have $\tan B\,O\,E = \frac{B\,E}{B\,O}$.

Substituting the above equivalents we have $\tan \frac{1}{2}\,I = \frac{T}{R}$, whence

$$T = R \tan \tfrac{1}{2}\,I. \qquad \textbf{(91.)}$$

In our example $R = 573.7$ ft.; $\frac{1}{2}\,I = 20°$; $\tan 20° = .36397$. $573.7 \times .36397 = 208.81$ ft. Measure back from the point E on both tangents the distance 208.81 ft. to the points B and C. Drive plugs flush with the ground at both points and set accurate center points, marked by tacks, in both. Directly opposite each of these plugs drive a stake called a **guard stake,** because it guards or rather indicates where the plug is. The stake at B, if the numbering of the stations runs from B towards C, will be marked P. C., and the stake at C marked P. T.

1252. To Lay Out a Curve With a Transit.—Having set the tangent points B and C, Fig. 282, set up the transit at B, the P. C. Set the vernier at zero and sight to E, the intersection point. Suppose B to be an even or "full station," say 18, and that it has been decided to set stakes at each hundred feet. Let the central angle $B\,O\,G$, measured by the 100-feet chord $B\,G$, be 10°; then, the deflection angle $E\,B\,G$, whose vertex B is in the circumference and subtended by the same chord *B G*, will be ½ *B O G* or 5°. Turn an angle of 5° from *B*, which in this case will be to the right; measure a full chain, 100 feet, from *B* and line in the flag at *G*; drive a stake at *G*, which will be marked 19. Turn off an additional 5° making 10° from zero, and at the end of another chain, at *H*, set a stake marked 20. Continue turning deflections of 5° until 20° or one-half of the intersection angle is reached. This last deflection, if the work has been correctly done, will bring the head chainman to the point of tangent *C*. It is but rarely that the P. C. comes at a full station. When the P. C. comes between full stations it is called a **sub-station,** and the chord between it and the next full station is called a **sub-chord.** Had the P. C. of the curve come at the sub-station, say $17 + 32$, the deflection for the sub-chord of $100 - 32$ or 68 feet, the distance to the next station, is found as follows: The deflection for a full station, i. e., 100 feet, is $5° = 300'$, and the deflection for 1 foot is $\frac{300'}{100} = 3'$, and for 68 feet the deflection will be $68 \times 3 = 204' = 3° 24'$, which is turned off from zero and a stake set on line, 68 feet from the transit, at Station 18. The length of a curve uniting two given tangents whose intersection angle is determined, is found as follows:

Suppose $I = 32° 40'$, and that the tangents are to be united by a 6° curve; 32° 40′ reduced to the decimal form is 32.666°; as each central angle of 6° will subtend a 100-foot chord, or one chain, there will be as many such chords or chains as 6 is contained times in 32.666, which is 5.444, that is, there will be 5.444 chains in the curve, or 544.4 feet, which is the required length of the curve. The P. C. and P. T. having been set and the station of the P. C. determined by actual measurement, say $58 + 71$, the station of the P. T. is found by adding to $58 + 71$, the station of the P. C., the calculated length of the curve, 544.4 feet. $58 + 71 + 544.4 = 64 + 15.4$, the station of the P. T.

Another method of calculation is the following: The sum of all the deflection angles is equal to one-half the intersection angle. The intersection angle being 32° 40′, one-half equals 16° 20′, which, reduced to minutes, equals 980′. The deflection for 100 feet is $\frac{6}{2}^\circ = 3^\circ = 180'$, and the deflection for 1 foot is $\frac{180'}{100} = 1.8'$; then, 980′, the total deflection, divided by 1.8′, gives 544.4 feet, the required length of the curve.

EXAMPLES FOR PRACTICE.

In the following examples, let I = angle of intersection, T = tangent, and L = length of curve.

1. $I = 16^\circ\ 13'$, degree of curve $= 3^\circ$, required, T and L.

Ans. $\begin{cases} T = 272.13 \text{ ft.} \\ L = 540.55 \text{ ft.} \end{cases}$

2. $I = 59^\circ\ 20'$, degree of curve $= 8^\circ\ 30'$, required, T and L.

Ans. $\begin{cases} T = 384.32 \text{ ft.} \\ L = 698.04 \text{ ft.} \end{cases}$

3. $I = 21^\circ\ 35'$, degree of curve $= 4^\circ\ 15'$, required, T and L.

Ans. $\begin{cases} T = 257.03 \text{ ft.} \\ L = 507.84 \text{ ft.} \end{cases}$

4. The degree of a curve is 5° 30′; what is the deflection angle for a chord of 16.2 feet? Ans. 26.7′.

5. The degree of a curve is 7° 15′; what is the deflection angle for a chord of 38.4 feet? Ans. 1° 23½′.

1253. Obstructions in the Line of Curve.—Frequently it happens that the entire curve can not be run in from the P. C. on account of obstructions. This is especially the case in either hilly or wooded country, and the transit has to "move up" to an intermediate point. For example, in Fig. 282, we will suppose that Station *H*, 200 feet from *B*, is the last point which can be set from the P. C. at *B*. A plug is driven at *H* flush with the ground and carefully centered, and a tack driven at the point. The deflection angle *E B H* is 10° to the right. The transit is set up at *H*, an angle of 10° to the left is laid off from zero, and the vernier clamped. The instrument is then sighted to a flag at *B*, the spindle clamped, and a close sight to the flag taken, the lower tangent screw being used to adjust the sight. The vernier clamp is then loosened and the vernier set at zero. The line of sight will then be on a tangent to the curve at *H*, and the deflection angles to *K* and *C* can be laid off as before and the stations *K* and *C* located.

1254. Tangent and Chord Deflections.—Let *A B* in Fig. 283 be a tangent, and *B C E H* a curve commencing at *B*. Produce the tangent *A B* to the point *D*. The line *C D* is a **tangent deflection,** and is the perpendicular distance from the tangent to the curve. If the chord *B C* be produced to the point *G*, making $C G = B C = C E$, the distance *G E* is a **chord deflection** and is double the tangent deflection *D C*.

1255. Given the radius $B O = R$, Fig. 283, to find the chord deflection *E G* and the tangent deflection $C D = F E$.

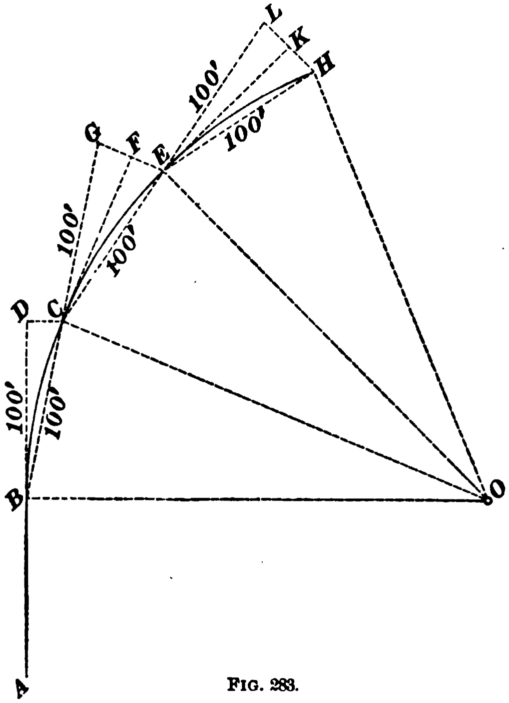


FIG. 283.

The triangles *O C E* and *C E G* are similar, since both are isosceles, and the angle $G C E =$ angle $C O E$. Hence, we have $O C : C E :: C E : E G$. Denoting the chord $C E$ by c and the chord deflection $E G$ by d, we have, from the above proportion, $R : c :: c : d$. Therefore,

$$d = \frac{c^2}{R}. \qquad (92.)$$

To find the tangent deflection, draw $C F$ to the middle point of $E G$. By Art. **1254**, $F E = D C =$ the tangent deflection. Hence, tangent deflection = one-half the chord deflection, from which

$$\text{tangent deflection} = \frac{c^2}{2R}. \qquad (93.)$$

1256. Practical Method of Determining Tangent and Chord Deflections.—Let it be remembered for a basis of calculation that the chord deflection for a one-degree curve, the chord being 100 feet in length, is 1.745 feet; for a 2° curve, double the deflection for a 1° curve, or 3.49 feet, and so on. The tangent deflection being one-half the chord deflection, for a 1° curve it will be .873 foot, for a 2° curve it will be 1.745 feet, etc.

Distances measured either on chords or tangents are expressed in decimal parts of a station, which is 100 feet, and

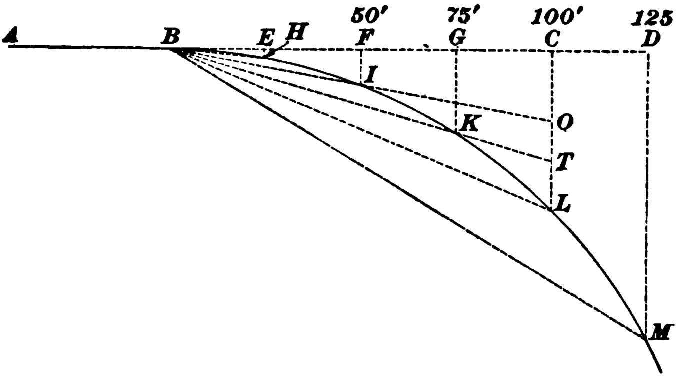


FIG. 284.

is assumed as 1. Thus, the tangent deflection for 75 feet will be expressed as the tangent deflection for .75 of a station. This expression is, however, confined entirely to the calculation, and is *spoken of* as the deflection for 75 feet.

The tangent to a curve is fixed in direction, being perpendicular to the radius at the point of tangency. Consequently, in a curve of given radius, the value of the tangent deflection depends wholly upon the length of the chord. Formula **93** shows that the tangent deflection is proportional to the square of the chord. Hence, knowing the tangent deflection for a chord of 100 feet, the tangent deflection for a chord of any other length can be found by the

Rule.—*Multiply the tangent deflection for a chord of 100 feet by the square of the given chord expressed as a decimal.*

Let $B\ K\ M$, Fig. 284, represent a 2° curve. For the chord $B\ L$ of 100 feet, the tangent deflection is 1.745 feet. Hence, for the chords $B\ I = 50$ ft., $B\ K = 75$ ft., and $B\ M = 125$ ft., the tangent deflections are, respectively, $.50^2 \times 1.745 = .436$ ft., $.75^2 \times 1.745 = .982$ ft., and $1.25^2 \times 1.745 = 2.727$ ft.

It is here assumed that the chords and corresponding tangents are of equal lengths. This is not strictly true, but is near enough when the degree of curve is small.

The above principle does not apply to chord deflections, however. The point G, Fig. 283, is in the prolongation of the chord $B\ C$, and the value of the chord deflection $G\ E$ is affected by the direction, and consequently, by the length, of $B\ C$. In the triangle $G\ C\ E$, the angle $G\ C\ F = \frac{1}{2}\ B\ O\ C$ and $F\ C\ E = \frac{1}{2}\ C\ O\ E$. Consequently, the triangle $G\ C\ E$ can be isosceles and similar to $C\ O\ E$ only when the angle $C\ O\ E = B\ O\ C$, that is, when $B\ C = C\ E$. Hence, formula **92** applies only *when the two chords preceding the station considered are of equal length.* When these chords are of different lengths, the chord deflection will be given closely by formula **92** if $\frac{1}{2}\ c\ (c + c')$ is substituted for c^2, where c' is the length of the second chord preceding the station. Or, if the *tangent* deflection f has been computed, the chord deflection d_0 will be given closely by the formula

$$d_0 = f\left(1 + \frac{c'}{c}\right). \qquad (\mathbf{92}a.)$$

1257. Laying Out Curves Without a Transit.—During construction, the engineer is often called upon to restore center stakes on a curve when the transit is not at hand. This can be accomplished reasonably well with a tape, as described in the following example.

In Fig. 285, *A B* is a tangent and *B*, at Sta. 8 + 25, is the *P. C.* of a 4° curve; a stake is required at each full station. The stakes at *A* and *B* are restored, determining the *P. C.* and the direction of the tangent. For a 4° curve the regular chord deflection for 100 feet is $4 \times 1.745 = 6.98$ ft., and the tangent deflection is 3.49 ft.

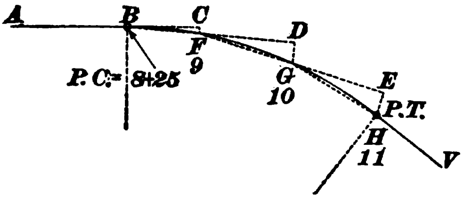


FIG. 285.

The distance from the *P. C.* to the next station *C* is 75 ft.; hence, the tangent deflection $CF = .75^2 \times 3.49 = 1.96$ ft. (Art. **1256.**) The point *F* is found by first measuring 75 feet from *B*, thus locating the point *C*, in the line with *A B*, then from *C* measuring $CF = 1.96$ feet, at right angles to *B C*; the point *F* thus determined will be Station 9. Next the chord *B F* is prolonged 100 feet to *D*; as *B F* is only 75 feet, formula **92***a* gives for *D G* the value $3.49 \times (1 + \frac{75}{100}) = 6.11$ feet. This distance is measured at right angles to *B D*; the point *G* thus determined will be Station 10. The position of Station 11, the *P. T.*, is determined in the same manner, except that, as the chords *F G* and *G H* are each 100 feet long, the regular chord deflection of 6.98 feet is used for *E H*. A stake is driven at each station thus located. Although a chord deflection is not at right angles to the chord theoretically, yet the deflection is so small, as compared with the length of the chord, that for curves of ordinary degree it is usually measured at right angles.

EXAMPLES FOR PRACTICE.

1. In a 5° curve, what are the tangent and chord deflections for a chord of 67 ft. following one of 100 ft.? Ans. $\begin{cases} f = 1.958 \text{ ft.} \\ d = 4.880 \text{ ft.} \end{cases}$

2. In a 7° 30′ curve, what are the tangent and chord deflections for a chord of 23.5 ft. following one of 100 ft.? Ans. $\begin{cases} f = .361 \text{ ft.} \\ d = 1.897 \text{ ft.} \end{cases}$

3. In a 6° 15′ curve, what are the tangent and chord deflections for a chord of 100 ft. following one of 84 ft.? Ans. $\begin{cases} f = 5.451 \text{ ft.} \\ d = 10.030 \text{ ft.} \end{cases}$

1258. To Determine Degree of Curve by Measuring a Middle Ordinate.—In track work, it is often necessary to know the degree of a curve when no transit is available for measuring it. The degree can be found by measuring the middle ordinate of any convenient chord, and multiplying its length by 8, which will give the chord deflection for that curve.

Let *A B*, in Fig. 286, be a 50-foot chord, measured on the track, and let the middle ordinate *a b* be .44 ft. $.44 \times 8 = 3.52$ = chord deflection for 50′, which, expressed in decimal part of a full station, is .5; $.5^2 = .25$. The chord deflection for 100 feet multiplied by .25 = the chord deflection for 50 feet, which we know by calculation to be 3.52 feet. Hence, $3.52 \div .25 = 14.08$ ft., the chord deflection for 100 feet, which, divided by 1.745, the chord deflection for a 1° curve, gives a quotient of 8.07, nearly. The inference is that the curve is 8°.

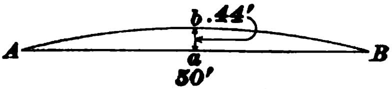


Fig. 286.

EXAMPLES FOR PRACTICE.

1. Length of chord is 50 ft., middle ordinate .35 ft.; required, degree of curve. Ans. 6° 25.08′.

(The original curve probably 6° 30′.)

2. Length of chord 40 ft., middle ordinate .21 ft.; required, degree of curve. Ans. 6° 1.02′.

(The original curve probably 6°.)

3. Length of chord 25 ft., middle ordinate .22 ft.; required, degree of curve. Ans. 16° 8.28′.

(The original curve probably 16°.)

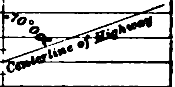
1259. Field Books.—To facilitate the field work of the engineer, **field books** have been published. They are portable, being carried in the pocket, and contain, in condensed form, general directions for the conduct of field work, together with all the necessary data in the form of tables, for prosecuting such work with accuracy and dispatch.

One of the first published in America is the work of John B. Henck, to whom most American engineers are under obligation.

1260. Note Books.—Various styles of note books are published, the pages being ruled to suit the particular kind of work being done. They are of three classes, viz., transit, level, and topography books. The latter are ruled in squares, which may be given any desired scale and greatly facilitate the accurate platting of topography in the field.

1261. How to Keep Transit Notes.—

A good form for location notes is the following:

Station.	*Deflection.*	*Tot. Angle.*	*Mag. Bearing.*	*Ded. Bearing.*	*Remarks.*	*June 20. 1894*
9						
8						
7						
6+95	*4°54' P.T.*	*15°00'*	*N. 35°30' E.*	*N. 35°15' E.*		
6+50	*4°00'*					
6	*3°00'*					 *70°00'* *Centerline of Highway*
5+50	*2°00'*				*5+80*	
5	*1°00'*				*5+60*	
4+50	*2°36'*	*5°12'*				
4	*1°36'*				*Int. Angle-15°00'*	*4° Curve R^t*
3+50	*0°36'*				*T-188.61 ft.*	*Def. Angle for 50 ft-1°00'*
3+20	*P.C. 4° R^t*				*P.C-3+20*	*Def. Angle for 1 ft-1.2'*
3					*Length of Curve-375 ft.*	
2					*P.T-6+95*	
1						
0	—		*N. 20°15' E.*	*N. 20°15' E.*		

In the first column the station numbers are recorded. In the second column are recorded the deflections with the abbreviations P. C. and P. T., together with the degree of curve and the abbreviation *R^t* or *L^t*, according as the line curves to the right or left. At each transit point on the curve, the total or central angle from the P. C. to that point is calculated and recorded in the third column. This total angle is double the deflection angle between the P. C. and the transit point. In the above notes, there is but one intermediate transit point between the P. C. and the P. T. The deflection from the P. C. at Sta. 3 + 20 to the intermediate transit point at Sta. 4 + 50 is 2° 36′. The total angle is double this deflection, or 5° 12′, which is recorded on the same line in the third column. The record of total angles at once indicates the stations at which transit points are placed. The total angle at the P. T. will be the same as the angle of intersection, if the work is correct. When the curve is finished, the transit is set up at the P. T., and the bearing of the forward tangent taken, which affords an additional check upon the previous calculations. The magnetic bearing is recorded in the fourth column, and the deduced or calculated bearing is recorded in the fifth column.

1262. Preservation of Notes and Records.—Notes should *never be erased.* If, on account of error or change of plan, they should cease to be of any value, they are crossed out, i. e., two diagonals are drawn across the page. All notes of permanent location should be copied each day into a separate book for office reference, to prevent confusion, and for record in case the original notes should be lost.

LEVELING.

1263. A Level Surface.—A level surface is one parallel to the surface of standing water. A water surface, though not theoretically level, owing to the curvature of the earth's surface, is assumed to be level and perpendicular to a vertical line, or the line of gravity.

The height of a point is its distance above a given level surface, measured on a vertical line, and is called its **elevation.** The process by which the elevation of a point is determined is called **leveling.**

1264. The Three Processes of Determining Elevations.

They are : 1st. By direct leveling.
2d. By indirect leveling ; and
3d. By barometric leveling.

1265. Direct Leveling.—In the process of direct leveling, a level line either actual or visual is prolonged so as to pass directly over or under the given point whose elevation is required. The elevation of any other point being determined in the same way, the difference in the elevations of the two points is found by subtracting the elevation of the lower from the elevation of the higher.

1266. Indirect Leveling.—In the process of indirect leveling, elevations are determined by means of lines and angles.

1267. Barometric Leveling.—In barometric leveling the elevation of a point is determined by the weight of the atmosphere at that point as registered by a barometer. The second and third processes will be explained later.

DIRECT LEVELING.

1268. General Principles.—Direct leveling depends upon three principles, two of which have already been stated, viz.: First, that the surface of a liquid in repose is level; second, that a vertical line is perpendicular to that surface, and, third, that a bubble of air confined in a vessel otherwise filled with liquid will rise to the highest point of that liquid. A common application of the third principle is seen in the spirit level used by carpenters and the level board used by masons.

1269. The "Y" Level.—There are a great variety of instruments for determining elevations. The one in most general use is the "**Y**" level, shown in Fig. 287.

This instrument consists of an **erecting telescope** *A B*, i. e., one which shows the image of the object to which the telescope is directed in its erect or natural position, resting in **Y**-shaped supports *C* and *D*, from which it takes its name.

The line of sight, or collimation, is identical to that in the transit explained in Art. **1225**, and is parallel to the level *E F*. The tube containing the eyepiece *G* has an exterior ring *H*, which is milled to assist the hand in turning the tube. This movement adjusts the eyepiece to the cross-hairs. The object glass at *B* is moved in or out by the milled headed screw *K*; *L* and *M* are parallel plates; the bar *O P* supports the **Y**'s and revolves on a spindle which is clamped by the screw *N*. By means of the tangent screw *X*, the telescope can be slowly turned horizontally. The telescope is leveled by means of the leveling screws *V*, *Q*, *R*, and *S*. The level is supported by the tripod *T*. The cross-hairs are of either platinum wire or spider threads, and are fastened to a ring which is held in place by capstan screws shown at *U*, and their movements are regulated in the same way as the movements of the cross-hairs of the transit explained in Art. **1225**.

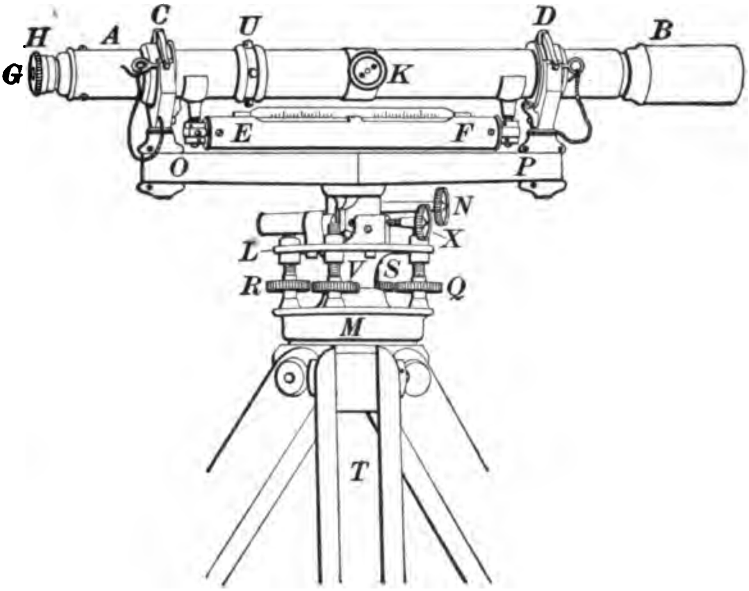


FIG. 287.

1270. The Bubble Tube.—The bubble tube is of glass bent upwards and so nearly filled with alcohol that only a bubble of air remains, which is always at the highest point in the tube. This tube is protected by a brass case, which is fastened to the underside of the telescope, and provided with the means for adjustment. The one end may be raised or lowered and the other end moved horizontally. Through a slit in the upper side of the case, the bubble tube is seen. Directly over it is a scale graduated in both directions from zero, which is over the center of the tube.

The **Y**'s *C* and *D* support the telescope, which is held in place by hinged clasps, or **clips,** as they are called, fastened by carefully turned pins, by means of which the telescope can be firmly held in any desired position. The **Y**'s rest upon the bar *O P*, to which they are fastened by lock-nuts, the one above, the other below, the bar, for raising or lowering. The bar revolves upon a finely turned steel spindle, resting in a socket of bell metal. The parallel plates *L* and *M* are united by a ball-and-socket joint, and held in place by the leveling screws *V*, *Q*, *R*, and *S*.

1271. Adjustments.—The first thing to do in preparation for actual leveling is to make the adjustments of the instrument.

There are three adjustments, as follows:

1. To make the line of collimation parallel to the bottoms of the collars, or rings, on which the telescope rests.

2. To make the plane of the level parallel to the line of collimation, or to the bottom of the collars.

3. To cause the bubble to remain in the center of the tube while the telescope is being revolved horizontally.

1272. First Adjustment.—To make the line of collimation parallel with the bottoms of the collars.

Plant the tripod firmly; choose some distant and clearly defined point, the more distant the better so long as the sight is distinct. Remove the pins from the clips and clamp the spindle, bringing the intersection of the cross-hairs to exactly bear on the point by means of the tangent screw. Revolve or turn the telescope on its supports through one-half a revolution, i. e., until it is bottom side up. If the intersection of the cross-hairs is still on the point of sight, it proves that the line of collimation is parallel to the bottoms of the collars. If, however, the line of sight is no longer on the point, move the cross-hairs by means of the capstan headed screws over one-half the space which measures the apparent error, being careful to move them in the opposite direction to that in which it would appear they should be moved. The apparent error is double the real error, and is explained in Fig. 288.

Let the instrument stand at *A* and sight to the point *B*, and suppose that when the telescope has been revolved half way around, the point *B* appears to be at *C*, then will the distance *B C* be double the real error, and the true line of sight will be at *D*, half way between *B* and *C*. Sometimes both cross-hairs are out of adjustment and they must be moved alternately until the intersection of the cross-hairs, i. e., the line of collimation, will pass through the same point throughout a complete revolution of the telescope.

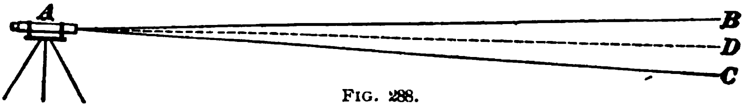


FIG. 288.

1273. Second Adjustment.—The second adjustment is to make the plane of the level parallel to the line of collimation, or to the bottoms of the collars, and is made as follows:

Remove the pins and open the clips; place the telescope over a pair of leveling screws and clamp the spindle. Bring the bubble to the middle of the tube by means of the leveling screws, and revolve the telescope through an eighth of a revolution. The bubble tube will stand out at an angle with the **Y**'s. If the bubble runs it shows that a vertical plane passed through the longitudinal axis of the bubble tube is not parallel to a vertical plane passed through the line of collimation. To correct the error, bring the bubble nearly back by means of the check nuts which regulate the lateral movement of the tube, and repeat the operation until the bubble ceases to run while the partial revolution is made. To complete the bubble adjustment, level the telescope and take it out of the **Y**'s and turn it "end for end." If the bubble remains in the center of the tube, the second adjustment is complete. If it runs to one end, bring it half way back by means of the check nuts provided for raising or lowering one end, and the rest of the way, i. e., to the middle of the tube, by means of the leveling screws. Repeat the operation, as the adjustment can rarely be made with one trial.

1274. Third Adjustment, sometimes called the "Bar Adjustment."—This is to cause the bubble to remain in the center of the tube while the telescope is being revolved horizontally.

Level the instrument, using both sets of leveling screws. Having centered the bubble carefully with one pair of leveling screws, turn the telescope end for end directly over the same pair of leveling screws. If the bubble runs, bring it half way back by means of the locknuts at the end of the level bar and complete the leveling with the leveling screws. Repeat the operation, as two or three trials will probably be necessary to complete the adjustment, so that the bubble will remain in the center of the tube throughout an entire horizontal revolution of the telescope.

The adjustments of the level should be tested every day when in constant use, as any defect in them will detract from the value of the work done, and a serious defect will necessitate a repetition of the work.

The cross-hairs are placed at right angles to each other, one of which should be vertical and indicate to the leveler whether the leveling rod is being held plumb. If the vertical cross-hair is "out of plumb," adjust it by loosening the capstan screws which hold the ring, to which the cross-hairs are fastened. Suspend a plumb-bob at a suitable distance from the level, and having sighted to it, tap the capstan screws sufficiently hard to cause the cross-hairs to move. In this way the vertical hair can be made to coincide with the plumb line, which is a true vertical.

1275. Sensibility.—The sensibility of the level depends directly upon the radius of the curve of the bubble tube.

The graduated scale placed directly over the bubble tube measures the movement of the bubble. The sensibility of the level may be determined as follows: Having leveled the instrument, take a reading on the rod held say 200 feet from the instrument. Suppose this reading to be 5.61 feet; with the leveling screws cause the bubble to move over one division of the scale. Suppose the rod then reads 5.63 feet. Denote the radius of the bubble by x, Fig. 289, the distance of the rod from the instrument by d, the difference of rod readings by h, and the movement of the bubble by S. From the approximately similar triangles we have $h : S :: d : x$, or $.02 : .01 :: 200 : x$, whence $x = \frac{2.00}{.02} = 100$ feet, the radius of the bubble tube.

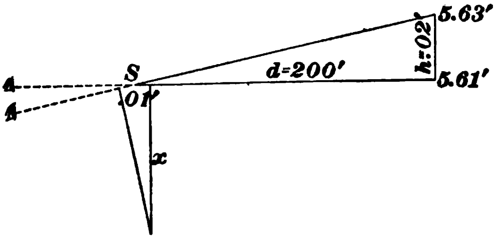


FIG. 289.

1276. Use and Care of the Level.—The level should not be used in rainy weather if it can be avoided. Moisture obscures the lenses and is otherwise injurious to the instrument. When rain is unavoidable, wipe the lenses frequently with a soft linen handkerchief, and when returned to the office or camp, thoroughly wipe, finishing with a piece of dry chamois skin and place in a warm, dry place so that every particle of moisture may be removed. Never carry the level with the spindle clamped. This rule is especially important when working in a wooded country where underbrush is dense. When *unclamped*, the level turns freely upon the spindle and yields readily to any pressure. A blow which would inflict no injury upon an *unclamped* instrument might seriously damage one while *clamped* and *rigid*.

1277. Power and Definition.—The power of a telescope is measured by the apparent nearness to which the image of the object is brought to the eye of the observer.

The definition of a telescope is measured by the degree of clearness of the outline of the image.

1278. Target Rods.—Target rods are divided into two classes, viz., those which are self-reading, or speaking rods, and those which are not self-reading. Railroad work is done chiefly with a self-reading rod. That in most general use is called the Philadelphia rod, and is shown in Fig. 290. It is in two sections held together with brass clamps *A* and *B*, one section sliding over the other. When closed, the rod measures 7 feet, sliding to 12 feet. It is graduated to feet, tenths, and hundredths. The feet are marked in large red figures, half above and half below the marks of division; tenths of feet are marked in black figures from 1 to 9, the lines of division reaching half way across the face of the rod; hundredths are marked by lines $\frac{1}{100}$ of a foot in width, alternating white and black, and extending about one-third the way across the face of the rod. The target is either circular or elliptical and divided into quarters, alternating red and white. The division lines are so arranged that when the rod is held in a vertical position one of them will be horizontal and the other vertical. The target *C* is fastened to a collar which slides up and down the rod, and is

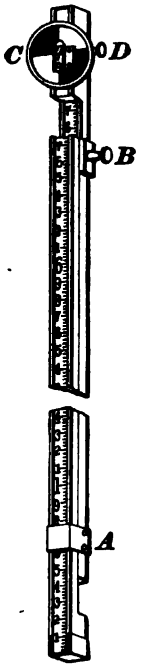


FIG. 290.

fitted with a screw *D*, which clamps it at any desired point. An opening more than one-tenth of a foot in length is cut in the face of the target. A vernier is fastened to the target whose zero point exactly coincides with the line which divides the target horizontally. It lies within the opening, on the face of the rod, and reads to thousandths of a foot. To prevent wear, the foot of the rod is shod with brass. Rod readings under 7 feet are usually taken with the two sections closed, and the target moved up or down until the horizontal line on the target coincides with the horizontal cross-hair of the telescope. When readings of more than 7 feet are taken, the clamp at *B* is loosened and the sliding section moved upwards until the horizontal line of the target and the horizontal cross-hair of the telescope coincide. The rod is then clamped, and is called a **long** or **high rod,** and can be read to thousandths with the vernier attached to the collar at *B*. In setting the target, the leveler should read the rod as closely as he can with the level, calling the reading to the rodman, who sets the target at the given reading and holds the rod up for a **check reading.** Four times out of five the leveler's reading will be the correct one, even to thousandths. More mistakes are made in reading the number of feet than the number of tenths. The leveler by first calling the reading to the rodman will be certain to prevent such an error, as it would at once be detected in the check reading. An experienced rodman can hold a rod practically plumb, and for all ordinary work his care is considered sufficient. For work requiring the greatest possible accuracy, such as bridge foundations, a hand level, which fits closely to the angle of the rod and carries two small spirit levels, is used to accurately plumb it. In using a rod which is not self-reading, all readings are taken with the target.

1279. Examples in Direct Leveling.—The principles of direct leveling are illustrated in Fig. 291.

Let *A* be the starting point, which has a known elevation of 20 feet. The instrument is set at *B*, leveled up, and sighted to a rod held at *A*. The target being set, the reading, 8.42 feet, called a backsight, is the distance which the point where the line of collimation cuts the rod is above the point A, and is to be added to the elevation of the point A. $20.00 + 8.42 = 28.42$ is called the height of instrument and designated H. I. The instrument being turned in the opposite direction, a point C is chosen, which must be below the line of sight. This point is called a turning point, and is designated by the abbreviation T. P. Drive a peg at C or take for a turning point a point of rock or some other permanent object upon which the rod is held. The reading at this point is a foresight, and is to be subtracted from the height of the instrument at B to find the elevation of the point at C.

Let the rod reading be 1.20 ft. As this reading is a foresight, it must be subtracted from 28.42, the height of instrument at B; $28.42 - 1.20 = 27.22'$, the elevation of the point C. As the ground rises abruptly, the rodman should slide the rod to its full length, being careful to keep it on the same point C. The leveler carries the instrument to D, which should be of such a height above C that when

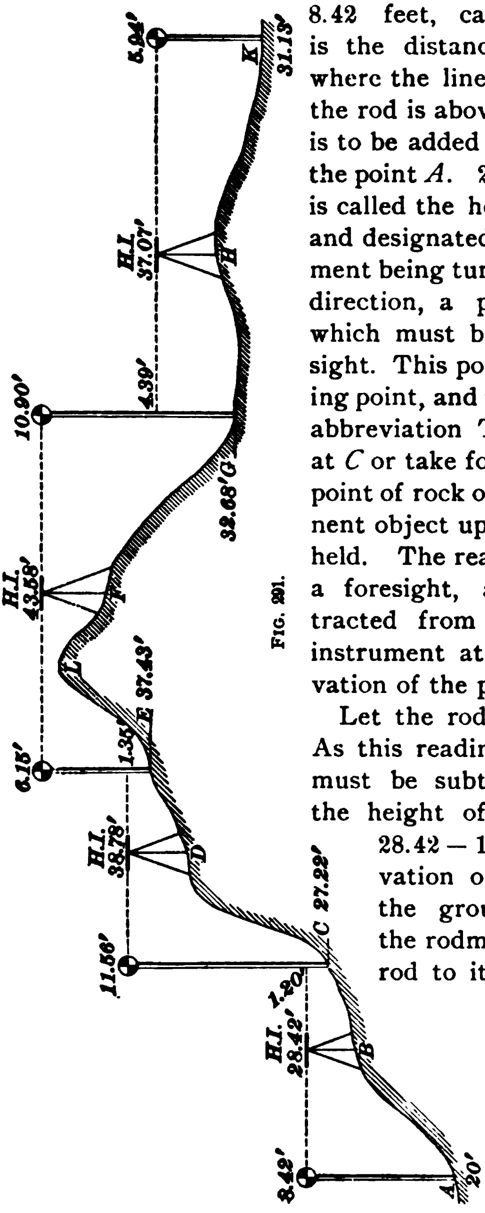


FIG. 201.

leveled up the line of sight will cut the rod near the top. The backsight to *C* gives a reading of 11.56 ft., which, added to 27.22 ft., the elevation of *C*, gives 38.78 ft., the height of the instrument at *D*. The rodman then goes to *E*, a point where a foresight reading is 1.35, which, subtracted from 38.78, the H. I. at *D*, gives 37.43 feet, the elevation of *E*. The level is then set up at *F*, being careful that the line of sight shall clear the hill at *L*. The backsight 6.15 ft. added to 37.43 ft., the elevation of *E*, gives 43.58 ft., the H. I. at *F*. The rod held at *G* gives a foresight of 10.90 ft., which, subtracted from 43.58, the H. I. at *F*, gives 32.68, the elevation at *G*. Again moving the level to *H*, the backsight to *G* of 4.39 ft. added to 32.68, the elevation of *G*, gives 37.07 ft., the H. I. at *H*. Holding the rod at *K* a foresight of 5.94 subtracted from 37.07 gives 31.13, the elevation of the point *K*. The elevation of the starting point *A* is 20.00 ft.; the elevation of the point *K* is found by direct leveling to be 31.13 ft., and the difference in the elevations of *A* and *K* is $31.13 - 20.00 = 11.13$ ft.; that is, the point *K* is 11.13 feet higher than the point *A*.

1280. A Datum Line.—A **datum line** is the base line to which the elevation of every point of a series is referred. Thus, in Fig. 291, the datum line or plane is 20 feet lower than the point *A*, and the elevations of the points *A*, *B*, *C*, *D*... *K* are their elevations above this datum line. Such a series of elevations is called a line of levels.

1281. Turning Points.—**Turning points,** mentioned in Art. **1279,** are the points where backsights and foresights are taken. The backsights are plus (+) readings, and are to be added; the foresights are minus (−) readings, and are to be subtracted. The rodman should make a peg of well-seasoned oak, or other hard wood, about 9 inches in length, 1 inch in diameter, sharpened at one end and rounded at the other end, which is the turning point. For driving the peg he should carry in a leather scabbard a light hatchet. A point for a foresight having been determined, the rodman drives the peg firmly in the ground and holds the rod upon it. After the instrument is moved, set up, and a backsight taken, the peg is pulled up and carried in the pocket until another turning point is called for. Turning points should be taken at about equal distances from the instrument in order to equalize any small errors in adjustment. In smooth country an ordinary level will permit of sights of from 300 to 500 feet. A good rodman is as necessary to accurate and rapid leveling as a good leveler. A man who is inattentive to the work in hand, or averse to rapid movement, is not fit for either place. In most localities, a line of levels of any considerable length will have enough rough places in it, i. e., places where considerable changes in elevation occur, to retard progress, however diligent the level party may be. Laziness or carelessness merit immediate discharge, and usually receive it.

1282. Bench Marks.—On railroad surveys, permanent points called **bench marks** should be established at intervals of from 1,000 to 2,000 feet, depending upon the nature of the country. Any permanent object, such as a stone door sill, a tree, or point of large rock, will serve for a bench mark. Where trees are available, they are always used, the point being cut on a projecting root. On preliminary lines they should be as near to the line as possible. A tree with a large exposed root is chosen, the bench mark is cut into the root in the form of a pyramid, a tack is driven into the apex and the rod held upon it. The tree is blazed smooth and the letters B. M., together with the elevation of the mark, written with red chalk. A bench mark of this kind is shown in Fig. 292, the point being at *A* and the elevation recorded at *B*.

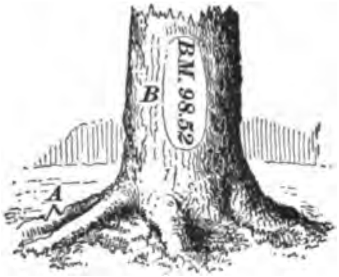


FIG. 292.

1283. Check Levels.—Check levels or test levels are taken for the purpose of checking and proving the accuracy of a line of levels before their adoption as a basis for construction. Usually intermediate points or stations are not taken, but only the turning points necessary to cover the line. Readings are taken at all the bench marks, and the correct elevations marked. The adjustments of the instrument should be frequently tested, and the rodman should carry a rod level to insure the plumbing of the rod.

1284. Water Checks.—When the line of survey follows the shore of a body of water having no current, such as a lake or pond, its surface can be used as a check, since its level for any ordinary space of time will remain unchanged. The sea, whose level is constant, is the base for all barometric leveling, and at all seaports for direct leveling.

1285. Rapid Work.—The rate of progress is limited by the transit party. If the country is open and rolling, where long sights are frequent and chaining easy, the level party will not keep up with the transit party. If the country is smooth and open, both parties can make about the same progress. If, however, the country is thickly covered with underbrush or heavy timber, the level party will have much idle time. A good day's work will vary, according to conditions, from three to eight miles.

The target should be set by signals given by the leveler. An upward movement of the hand is the signal for raising the target, and a downward movement the signal for lowering it; a circle described by the hand is the signal for clamping the target, and a wave with both hands indicates that the target is properly set.

All intermediate readings are read by the leveler, whose signal "All right" is a single outward wave of the hand, the rodman being careful to keep the rod at full length.

The rodman should always call out the rod reading, giving first the number of feet, or, if the reading is less than 1 foot, call the figure "naught," never "ought," then pausing a moment, call the decimal part of the reading. If the rod is being read to hundredths only, the number, 8.40, is read:

eight-four, naught; if 8.04, it is read: eight-naught, four. If the rod is to be read to thousandths, the number, 8.401, is read: eight-four, naught, one; if 8.410, it is read: eight-four, one, naught.

The distinctness of a call is in no way proportional to the amount of noise in it. A few days' practice will enable a rodman with moderate effort to call a reading so as to be distinctly heard at a distance of 500 feet. Should a high wind be blowing, the sights will be shorter, owing to the vibration of the instrument, and the rodman's work proportionally lessened. The rod reading should always be recorded before moving the instrument. The leveler may check the reading as he passes the rodman. In general, however, the leveler relies entirely upon the accuracy of the rodman's readings. If he can not be trusted, his place should at once be supplied by one who can be trusted.

In taking levels on preliminary railroad surveys, frequently the turning points, as well as intermediate stations, are read by the leveler without being checked by the target. The rodman has still plenty of occasion for the use of judgment, as the rate of progress depends largely upon the care shown in the selection of turning points.

1286. Sources of Error.—The principal sources of error are defects in adjustment, which are the fault of the leveler, and failure of the rodman to plumb the rod, and wrong target readings, which are the fault of the latter. Poor levelers and poor rodmen usually go in pairs. Haste or hurry are poor helps to progress. One can do rapid and accurate work without haste, but can not hurry and be either rapid or accurate.

The sun shining directly upon the object glass confuses the sight. To prevent this, most instruments are provided with a sun shade, which fits the end of the telescope, projecting over the object glass.

If the sun shade is lacking, the leveler can hold his hat so as to shade the object glass.

Wind is also a source of error, as it causes the instrument to vibrate, thus preventing the accurate setting of the target. The leveler should wait for a lull in the wind, during which, if his rodman is alert, he can get a close shot. At a second lull, he can check the target and feel safe in moving ahead.

Individual errors, called "personal equation," are defects in vision peculiar to the individual, so that two persons may set a target for the same rod, each giving a different reading; but as this personal equation, or error, is constant for the same person, it does not materially affect the accuracy of work.

1287. Necessary Degree of Accuracy.—In preliminary railroad work an error of .10 of a foot per mile is allowable. Time spent in reducing such inaccuracies is wasted. That painful degree of accuracy termed "hair-splitting" is no recommendation, and the gain in accuracy is more than balanced by increased cost and loss of time. It is a well-known fact that small inaccuracies tend to balance each other, and that a line of levels covering 20 miles, taken with a self-reading rod, will closely check a line taken with target readings and rod level.

1288. How to Keep Level Notes.—Forms for keeping level notes are various. One of the best forms, rarely or never seen in print, and yet one which is in general use among engineers, is shown on the following page:

The distinguishing feature of this form of level notes is a single column for all rod readings. The backsights being additive and the foresights subtractive readings, they are distinguished from other rod readings by the characteristic signs + and –. The turning points, whose foresight reading is –, are further designated by the abbreviation T. P.

1289. How to Check Level Notes.—There is one method of checking level notes which is in universal use. It provides for checking the elevations of turning points and heights of instrument only, which is sufficient, as all other elevations are deduced from them. The method is very simple and depends upon the fact that all backsights

1.

2.

Station.	Rod Reading.	Ht. Instrument.	Elevation.	Grade	Cut.	Fill.	Remarks.	Date.
B. M.	+ 5.61	105.61	100.00				On Root of White Oak	St'mp 10′ L. Sta. 0
0	6.10		99.50					
1	7.30		98.30					
2	8.40		97.20					
3	9.20		96.40					
T. P.	− 10.22		95.39					
	+ 5.41	100.80						
4	5.30		95.50					
5	4.20		96.60					
5 + 50	11.50		89.30				Spring Brook	
T. P.	− 2.52		98.28					
	+ 11.57	109.85						
6	6.20		103.60					
7	8.50		101.30					
8	10.10		99.70					
T. P.	− 11.53		98.32					

are additive or + quantities, and all foresights are subtractive or − quantities. The level notes described in Art. **1288** are checked as follows: The elevation of the bench mark at Station 0 is 100.00 feet, to which all backsights or + readings are to be added, and from this sum all foresights or − readings are to be subtracted. The sum of the + readings or backsights together with the elevation of the bench mark at 0 is 122.59. The sum of the − readings or foresights is 24.27, and the difference 98.32 feet is the elevation of the turning point last taken. As soon as a page of level notes is filled, the leveler should check them, placing a check mark √ at the last height of instrument or elevation checked. When the work of staking out or cross-sectioning is being done, the levels should be checked at each bench mark on the line. At the close of each day's work, the leveler must check on the nearest bench mark.

	+	−
Thus,	100.00	10.22
	5.61	2.52
	5.41	11.53
	11.57	24.27
	122.59	
	24.27	
	98.32	

1290. Profiles.—A profile represents a vertical section of the line of survey. In it all abrupt changes in elevation are clearly outlined. Vertical and horizontal measurements are usually represented by different scales. Irregularities of surface are thus rendered more distinct through exaggeration. For railroad work profiles are commonly made to the following scales, viz., horizontal, 400 feet = 1 inch; vertical, 20 feet = 1 inch.

A section of profile paper is shown in Fig. 293. Every fifth horizontal line and every tenth vertical line is heavy. By the aid of these heavy lines, distances and elevations are quickly and correctly estimated and the work of platting greatly facilitated. The level notes described in Art. **1288** are platted in Fig. 293. The elevation of some horizontal line is assumed. This elevation is, of course, referred to the datum line, and is the base from which the other elevations are estimated. Every tenth station number is written at the bottom of the sheet under the heavy vertical lines. The profile is first platted in pencil and then inked in black.

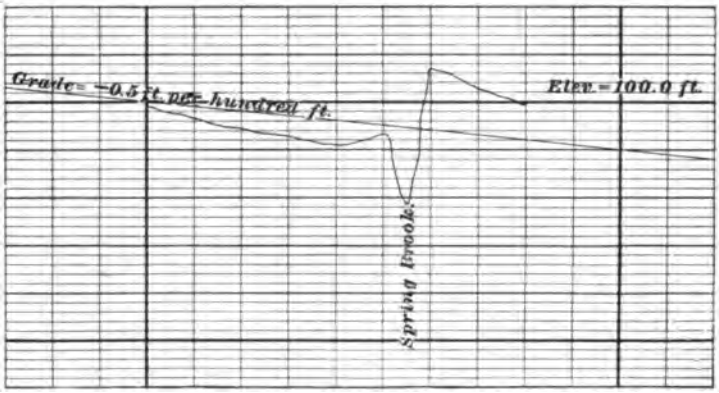


FIG. 293.

1291. Grade Lines.—The principal use of a profile is to enable the engineer to establish a **grade line,** i. e., a line showing the relative proportion of excavation and embankment in the proposed work. The **rate** of a grade line is measured by the vertical rise or fall in each hundred feet of its length, and is designated by the term per cent. Thus, a grade line which rises or falls 1 foot in each hundred feet of its length is called an ascending or descending 1 per cent. grade, and written + 1.0 or — 1.0 per hundred. A rise or fall of one-half foot in each hundred feet is called a five-tenths per cent. grade, and written + .5 or — .5 per hundred. The grade line having been decided upon, it is drawn in red ink.

EXAMPLE.—The elevation of Station 20 is 140.0 feet; between Stations 20 and 100 there is an ascending grade of .75 per cent.; what is the elevation of the grade at Station 71?

SOLUTION.—To obtain the elevation of the grade at Station 71, we add to the elevation of the grade at Station 20, 140 feet, the total rise in grade between Stations 20 and 71. Accordingly, $71 - 20 = 51$; .75 foot $\times$ $51 = 38.25$ feet; $140 + 38.25 = 178.25$ feet, the elevation of grade at Station 71.

TOPOGRAPHICAL SURVEYING.

1292. General Definition.—Topographical surveying is the location and representation of the inequalities of any portion of the earth's surface. The portion surveyed is conceived to be projected upon a horizontal plane, called a **plane of reference,** upon which all inequalities of surface as well as all conspicuous objects are shown in their true relative positions. The simplest and most generally used method of representing the **topography** of a given surface is by means of **contour lines.** A map containing an outline of a given surface, together with the contour lines representing its inequalities, is called a **contour map** of that surface.

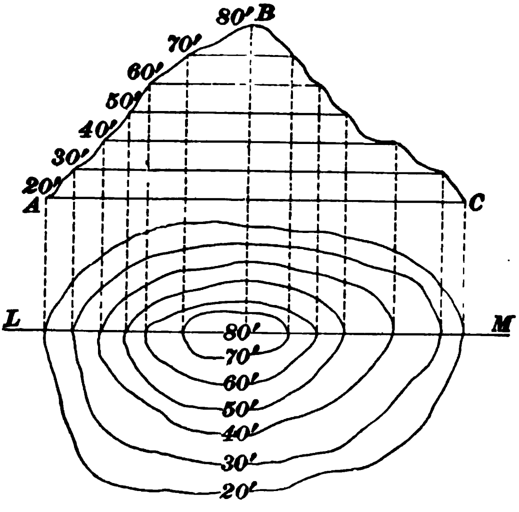


FIG. 294.

Let *A B C*, in Fig. 294, represent the outline of a hill, and suppose this hill to be gradually submerged in water, the water rising in successive heights of 10 feet. The flow, or shore line, at each successive rise is a **contour line.** The horizontal lines correspond to the surfaces of the successive elevations of the water. The points where these horizontal lines cut the edge of the hill are projected on the horizontal line *L M.* The irregular lines connecting the corresponding points of projection are contours. In Fig. 294 they are assumed to be 10 feet apart in vertical measurement.

1293. Conduct of a Topographical Survey.—The manner of conducting a topographical survey will depend upon the extent and outline of the surface and the degree of accuracy required. If the area be of comparatively regular dimensions, such as town or park sites, the usual practice is to lay out the area in squares. The lines of division are the bases for the location of all points within the area whose elevations are determined by direct leveling. If the area is long and narrow, as in a railroad survey, the line of survey is the base for the location of all points and for determining their elevations. Cross-sections of the surface are taken at suitable intervals, and changes in the slope of the surface are measured either by direct leveling or with a *clinometer* or *slope board.*

1294. The Hand Level.—The usual form, called the "Locke level," from the name of the inventor, is shown in Fig. 295. It consists of a brass tube *A B*, on the top of which is a spirit level *C*. In the lower part of the tube is a mirror which reflects the point at which the bubble should be when the instrument is level. A small hole *D* at one end and a cross-hair at the other give the level line. The observer holds the level to one eye, bringing it to a level line while he observes the object to which the level is directed with the other. In taking cross-sections with a Locke level, the following rule is recommended: The topographer has two or more assistants, three is the better number, a rodman and two tapemen. The rodman is provided with a rod at least 12 feet in length, of light weight, and of sufficient width to admit of large, distinct figures being painted upon

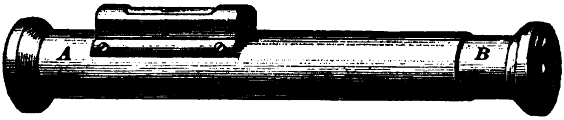


FIG. 295.

it, and divided to tenths of feet. The rod is painted like the Philadelphia rod; the face white, tenths of feet in black, and feet in red. Tapemen should use a tape 100 feet in length, of durable material. Chesterman with wire warp is best. The topographer first measures the distance of his eye above the ground, which is a constant quantity, to be subtracted from all the rod readings. He then stands at a station and keeps the rodman at right angles to the line of survey. The rodman, having reached the end of a slope, i. e., a point, where the rate of slope changes, he holds his rod at the point and the topographer takes the reading with the hand level. From this reading the topographer subtracts the constant, i. e., the height of his eye above the ground. The remainder is the difference between the elevation of the surface where the topographer stands and the surface where the rodman stands. The tapemen having measured the distance between the two points, the rate of slope is determined by dividing the distance measured by the difference in elevation. This method of taking slopes or cross-sections is illustrated in Fig. 296.

Let *A* be Station 156 of a preliminary survey. The topographer stands at *A*. The rodman goes to the point *B*, where the slope changes, holding his rod, which measures 16 feet in length, at that point. The topographer sights with his hand level and reads 7.5 feet on the rod. From this reading he mentally subtracts 5.3 ft., the height of his eye

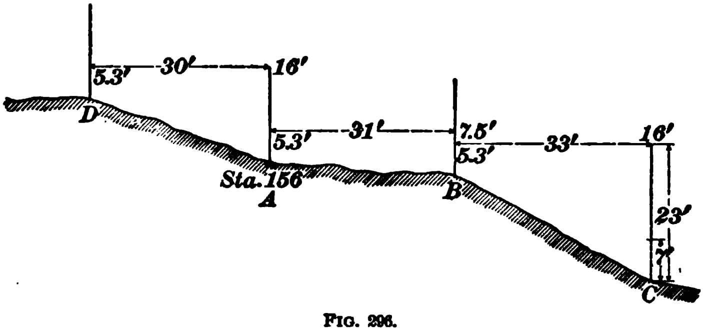


FIG. 296.

above the ground. The remainder, 2.2 ft., is the difference in elevation between the points *A* and *B*. Meanwhile, the tapemen find that the horizontal distance from *A* to *B* is 31 feet. The rate of the slope *A B* is the horizontal distance between the points *A* and *B*, 31 ft., divided by 2.2, their difference in elevation. The quotient is 14.1 and the slope is recorded $-\frac{2.2}{31}$. The topographer then moves to the point *B*, and the rodman goes to *C*, which is so much lower than *B* that with the rod held on the ground the line of sight will pass over the top of the rod. Here the rodman gives a "long" or "high" rod. Planting himself firmly at *C*, he raises the rod until the line of sight, from the topographer's eye, cuts the top of the rod, when the topographer calls "all right." He then notes where the bottom of the rod comes, and allows it to slide to the ground. Then adding to the length of the rod 16 ft., the distance from the ground to the point where the bottom of the rod came when the reading was taken, he calls out their sum to the topographer. In this example the rod is 16 feet and the addition 7 feet, making a high rod of 23 feet, which is common enough. The horizontal distance 33 feet, as measured by the tapeman, is also called out. The topographer makes the subtraction 5.3 from 23.0, and the difference 17.7 is written as the numerator of a fraction whose denominator is the horizontal distance 33. The slope being a descending one, the fraction will be $-\frac{17.7}{33}$, a slope of 1 to 1.9. In Fig. 296, the slopes *A B* and *B C* are **right slopes,** i. e., on the right side of the line of survey.

In taking the **left slopes,** the rodman and topographer change positions, the topographer going ahead and the rodman following. The topographer standing at *D* reads a rod of 16 feet held at *A*. Subtracting the constant 5.3, the remainder 10.7 is the difference between the elevations of *A* and *D* and is an ascending slope. The horizontal distance from *A* to *D* is 30 feet and the slope is recorded $+\frac{10.7}{30}$.

1295. Slope Angles.—Slopes are often measured with an instrument called a **clinometer,** which measures the angle which the line of slope makes with the horizontal, and is shown in Fig. 297. Tables are compiled giving the angle of slope and the horizontal distance for one foot of rise, as follows:

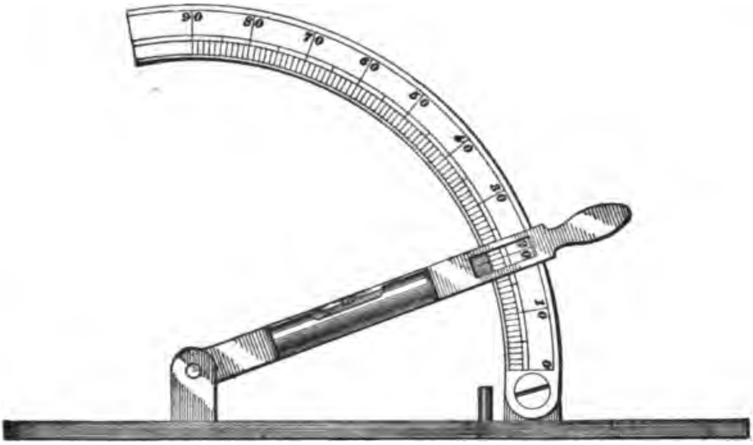


FIG. 297.

1° is 57.3 feet horizontal per 1 foot rise.
2° is 28.6 feet horizontal per 1 foot rise.
3° is 19.1 feet horizontal per 1 foot rise, etc.

1296. Platting Topography in the Field.—While some engineers favor the platting of contour maps in the field, the majority do not. To plat the map in the field, the topographer carries a case, the cover of which serves for a drawing board. The line of survey is divided into sections which are platted on different sheets, each sheet containing some of the immediately preceding section, so that by overlapping and pinning them together, a complete map of the line is obtained. The topographer carries in his case the sections covering his day's work, with the numbers and elevations of each station marked on the map. He pins a section to the cover of the case with thumb-tacks; his assistants measure the angle of the slope with a clinometer, together with the horizontal distance; and from the table of slopes which he carries, the topographer determines the location of the contours and sketches them on the map. A better practice is to measure and record the slopes, keeping as close to the transit party as possible, and provide an extra man to work up the notes in the office under the direction of the topographer.

1297. Eye Measurements.—Though practice will greatly aid the eye in estimating distances, yet it is not to be relied upon when anything like exactness is required. In taking slopes, the length of the last one only may be estimated by the eye. More distant objects which lie without the possible range of location may be sketched in with the aid of the eye alone.

1298. Form of Topographer's Notes.—A good form for a topographer's notes is shown in the accompanying diagram :

Station.	Lt. Line.		Rt. Line.
0	$+\frac{8.4}{45}$ $+\frac{10.0}{30}$	Center Line.	$-\frac{11.4}{35}$ $-\frac{8.8}{50}$ for 100′
1	for 60′ $+\frac{11.5}{40}$		$-\frac{11.0}{53}$ $-\frac{7.0}{50}$ for 100′
2	for 60′ $+\frac{10.3}{40}$		$-\frac{11.5}{50}$ $-\frac{6.8}{55}$ for 100′
3	Same as 2		Same as 2
4	for 50′ $+\frac{11.8}{40}$		$-\frac{10.5}{55}$ $-\frac{7.5}{40}$ for 100′
5	for 50′ $+\frac{12.0}{35}$		$-\frac{11.3}{54}$ $-\frac{6.8}{40}$ for 100′
6	for 60′ $+\frac{10.4}{40}$		$-\frac{10.5}{50}$ $-\frac{7.5}{43}$ for 100′

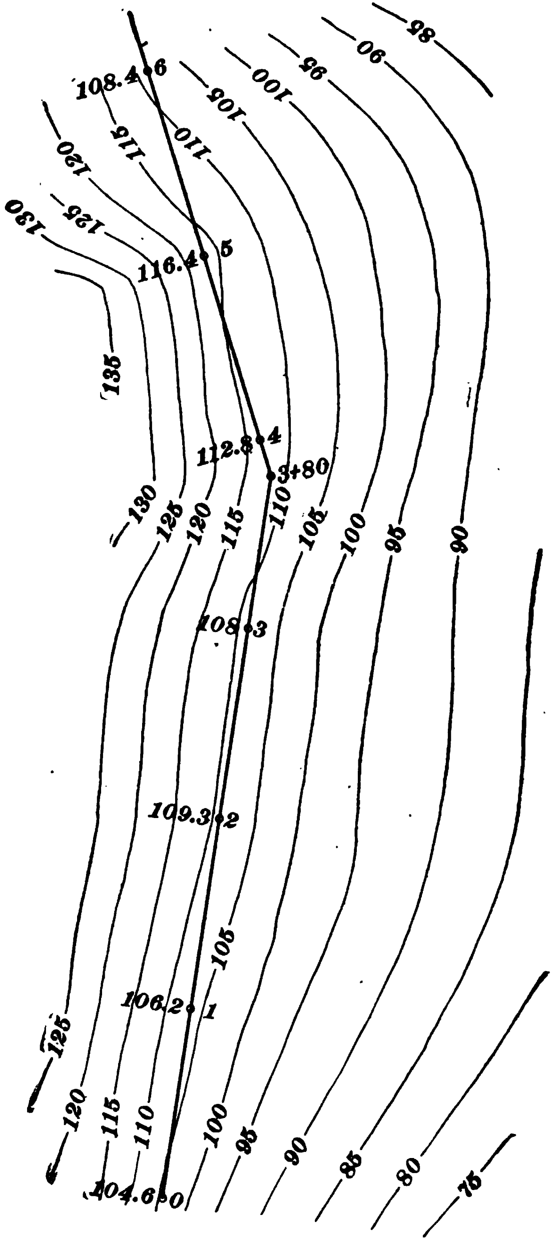

85
90
95
100
105
110
115
120
125
130
135
108.4
6
116.4
5
112.8
4
3+80
108
3
109.3
2
106.2
1
104.6
0
130
125
120
115
110
105
100
95
90
105
125
120
115
110
100
95
90
85
80
75

FIG. 298.

They are a record of the cross-sections or slopes of a preliminary railroad survey, the line of which extends along the side of a steep hill. The slopes are taken with a Locke level and rod, giving the actual differences in elevation between the points of change of slope. The alignment of the survey is shown in Fig. 298, and the contours are platted from the foregoing notes. The contours are 5 feet apart, i. e., the vertical rise between them is 5 feet.

The elevations of the stations the topographer has obtained from the leveler. The stations are marked on the plat, either by a dot, or, what is better, a dot enclosed in a small circle. The number of the station is marked at the right a little space ahead of the circle, the elevation on the left of the line and opposite to the number of the station. The cross-section lines are sometimes drawn on the map, very fine and at right angles to the center line, but usually the lines are omitted, the draftsman giving the true direction with his offset scale when locating the contours. In Fig. 298, the elevation of Station 0 is 104.6 feet. To reach the next contour above, viz., *105*, a rise of .4 foot must be made, and to reach the next lower contour a fall of 4.6 feet is necessary. From the notes, we find on the left of the line a rise of 10 feet in a horizontal distance of 30 feet or a rise of 1 foot in 3 feet, and for a rise of .4 foot we must go to the left of the line .4×3 = 1.2 feet to contour *105*. To reach contour *110*, which is 5 feet higher, we must go 5×3 = 15 feet farther to the left. This distance added to 1.2, the distance to contour *105*, gives 16.2 feet, the second offset. We find by adding 10 feet (the rise in going 30 feet to the left of the line) to 104.6 feet, the elevation of Station 0, we have 114.6 feet, which is the elevation of the end of the first slope. An additional rise of .4 foot must be made in order to reach contour *115*. The second slope is a rise of 8.4 feet in a distance of 45 feet, or a rate of 1 foot in 5.4 feet. Multiplying 5.4 feet by .4, we have 2.2 feet, which is to be added to 30 feet, to reach contour *115*, and gives a distance of 32.2 feet. Contour *120* will be 5.4 feet × 5 = 27.0 feet beyond contour *115*, or 59.2 feet from the center line

In the same way the contours to the right of the line are located. Tenths of feet are dropped in the computed distances, as they are too small for platting, and the nearest foot is taken.

Having located the contours by offsets for several consecutive stations, points of equal elevation are joined freehand, forming the contour lines, care being taken that lines of different elevation are kept distinct from each other and conforming to the curves and undulations of the original surface.

1299. Working Up Notes.—A good rule is to work up the notes for a considerable section before platting, thus avoiding the delay from continual change of work. The following form of working up notes is a good one, notes for each station being separated from those for other stations by a few strokes of a pencil. The example given is for Sta. 0, in Fig. 298.

Sta. 0. Elev. 104.6.

Rt. 14 feet to contour *100.*
Rt. 29 feet to contour *95.*
Rt. 53 feet to contour *90.*
Rt. 81 feet to contour *85.*
Rt. 110 feet to contour *80.*
Rt. 137 feet to contour *75.*

Lt. 1 foot to contour *105.*
Lt. 16 feet to contour *110.*
Lt. 32 feet to contour *115.*
Lt. 59 feet to contour *120.*

Contour lines are usually drawn first with pencil and afterwards inked in black. Short gaps are left in the lines at suitable intervals, in which their elevations are written. These should be of sufficient frequency to show at a glance the elevation of any contour.

Situations are continually recurring where the side slopes give but an inadequate idea of the topography. This is particularly true when the line of survey follows a stream with numerous tributaries and where highway crossings are frequent. In such cases the topographer will supplement the side slopes with free-hand sketches, which are invaluable helps in making topographical maps.

INDIRECT LEVELING.

1300. Indirect leveling is the process of determining elevations by either lines or angles or both. A common example in indirect leveling is given in Fig. 299.

Let DB be a flag-staff whose height is required. Set up a transit at A. Level carefully both the vernier plate and the telescope. The vertical arc will, if in adjustment, read at zero. Sight to C, the point where the horizontal line of sight strikes the flag-staff. Measure the distance $AC = 180$ feet, $CD = 4.2$ feet, and the diameter of the staff at $C = 1.5$ feet. Measure the vertical angle $CAB = 26° 10'$. From rule **5**, Art. **754**, we have $\tan A = \frac{CB}{AC + \frac{1}{2} \text{dia. staff}}$. One-half diameter staff at $C = .75$ foot. Substituting known values, we have $\tan 26° 10' = \frac{CB}{180.75}$, whence $CB = 180.75 \times \tan 26° 10' = 180.75 \times .49134 = 88.809$ feet; $88.809 + 4.2 = 93.009$ feet $= DB$, the height of the flag-staff.

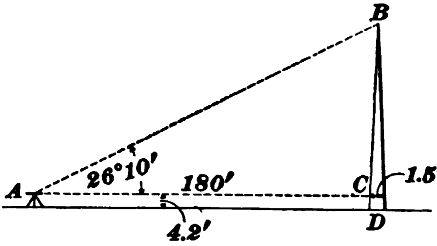


FIG. 299.

1301. Stadia Measurements.—The theory of the stadia is familiar to most engineers, yet comparatively few of them make any practical application of it, even when it would be greatly to their advantage.

In stadia work an ordinary leveling rod is generally used, and answers every purpose. It should be made of hard wood, such as mahogany, which is least affected by changes of temperature, and should be from 10 to 12 feet long, 2 inches wide, and about $1\frac{1}{4}$ inches thick. It is divided into feet, and each foot subdivided into tenths. The spaces corresponding to these latter divisions are painted alternately red and white, the number of tenths each space represents being painted in prominent black figures on the lines of division. The space directly below each footmark should be inlaid with a mirror to reflect the light and enable the surveyor to read the rod at long distances with greater precision. The rod should also be provided with a sliding target. The best instrument to employ in this class of work is a transit reading to 30″. Besides the horizontal and vertical cross-wires which appear in the field of view of the ordinary transit telescope, the stadia transit is provided with two additional horizontal wires placed parallel with the horizontal wire in the plain transit, and at an equal distance above and below it, as shown in Fig. 300. These two extra wires are so placed that, if the stadia rod is held at a point 100 feet distant from the telescope, they will enclose 1 foot of the length of the rod. For example, if the lower wire coincides with the 4-ft. division, and the upper wire with the 5-ft. division of the rod, the distance from the center of the instrument to the rod will be 100 ft. + the *constant* for the particular transit used. The starting point for stadia measurements is often indiscriminately assumed to be either the center of the instrument, the center of the cross-wires, or from a plumb line dropped from the object glass; but, owing to the deflection of the sight due to the action of the lenses, the precise starting point for stadia measurements is a point as far in front of the object glass as its focal length; for example, if the focal length of the object glass is 6 inches, the starting point is 6 inches in advance of a plumb line dropped from the object glass. The distance from this point to the center of the instrument is "constant" for the *same* instrument, and must be added to the recorded stadia distance at every sight. In making a stadia survey, the transit should first be tested. Having found as level a plane as possible, test and adjust the level so that the vertical arc will read zero when the telescope is in a perfectly horizontal position; measure off very carefully from the center of the instrument, the short distance equal to the constant of the instrument, say 1.25 feet; from this point accurately measure a

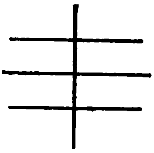


FIG. 300.

distance of 400 feet, driving a stake at each 100 feet. It is advisable to measure this test line with two or more steel tapes, and then take the average. As it will be necessary to test the cross-wires every few days, it is important that the test line should be conveniently located and very accurately measured. The line now measures 401.25 feet, as follows: First section, measuring from the center of the instrument, 101.25 feet, then three sections of 100 feet each, as shown in Fig. 301.

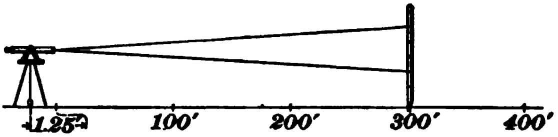


Fig. 301.

Direct the rodman to hold the rod on the point 201.25 feet from the instrument, and adjust the stadia wires so that they will include 2 feet on the rod. First adjust the upper to the center wire so as to include 1 foot, then adjust the lower to the center to include one foot. When this has been done, let the rod be held at the point 301.25 feet distant. The wires should now inclose 3 feet, 1.5 feet being included between the upper and center wires and 1.5 feet between the center and lower wires. Now test the point at the extremity of the line; the wires should at this distance include 4 feet. Instruct the rodman to hold the rod on the first point, 101.25 feet from the instrument, and if the stadia wires now include one foot, the instrument is in adjustment; if not, the operations must be repeated until the instrument reads correctly at every point. The ratio of the constant does not increase with the distance, but remains the same whether the distance of the sight be 50 or 2,500 feet.

At the beginning of a survey, the target on the rod is set at a height equal to that of the instrument, i. e., the distance from the ground-line to the axis of the telescope. This is done with the view of having the line of sight parallel with an imaginary line between the foot of the instrument and the foot of the rod, which gives the exact vertical angle or degree of slope between the instrument and rod and a perfectly level plane. The rod is now held on a point where a sight is desired, and the transitman turns the telescope until the center wire and the center line of the target coincide; see Fig. 302. He then clamps the telescope, and reads the angle of elevation or depression, as the case may be, on the vertical arc, which is say 10° 26′; and, if the rod is held on a point at a greater elevation than that of the telescope, this angle will be one of elevation, and he will record it thus, + 10° 26′; but if the rod is held on a point lower than the instrument, the telescope will be correspondingly depressed, and the angle is recorded thus, – 10° 26′. The distance on the rod intercepted by the stadia wires is read and recorded. Assuming that the lower wire coincides with the 3.5 ft. division line, and the upper one cuts the rod at 7.46 feet, the intercepted distance is 7.46 – 3.5 = 3.96 feet, and is thus recorded. The needle is next read, or, if it be an angular survey, the direction is platted and recorded. Having thus obtained the vertical angle, intercepted distance, and bearing, this sight is finished and the surveyor is ready to move to the next station.

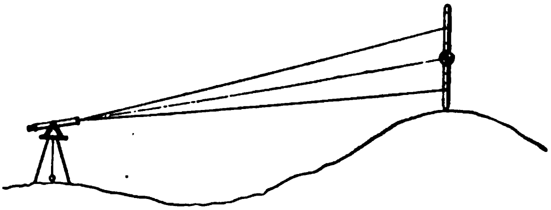


FIG. 302.

Before any platting can be done, the distances must be calculated and reduced to the horizontal. This may be accomplished by means of the table of Horizontal Distances and Differences of Elevation for Stadia Measurements. In using the tables, proceed as follows: Look for the vertical angle, in this instance 10° 26′, and under the head Hor. Dist. find the number 96.72. Then, this number multiplied by the distance intercepted by the stadia wires, viz., 3.96, equals 96.72 × 3.96 = 383.01; now, at the foot of the page, under 10° and opposite $c = 1.25$ (the constant of the instrument), find the corrected distance 1.23, which, added to 383.01, gives 384.24 feet, the corrected horizontal distance, which is recorded in the column provided for that purpose in the note book.

The difference of level is found thus: Under the head Diff. Elev., find 17.81, the number corresponding to the vertical angle 10° 26′. This number multiplied by the intercepted distance equals 17.81 × 3.96 = 70.53; at the foot of the column find .23, which, added to 70.53, gives 70.76 feet as the difference of elevation, and is recorded as such in its proper place. Proceed in the same manner to find the horizontal distances and differences of level of all the other points observed. The relative elevations of the various points observed, above or below any adopted datum line or plane of reference, can be readily determined by means of the signs + and – prefixed to each vertical angle recorded. Thus, assuming the survey to start from a B. M. 497.32 feet above the adopted plane of reference, and the first angle recorded to be, as before stated, + 10° 26′, corresponding to a difference of level of + 70.76 feet, the point observed will be 497.32 + 70.76 = 568.08 feet above the datum plane. Where, however, boundary lines only are being run, it is unnecessary to compute the levels, but the vertical angles must be recorded in all cases, in order to correct the distances.

The calculations may be made, without the use of tables, in the following manner:

To obtain the horizontal distance, the following formula is employed:

$$D = c \cos n + a\,k \cos^2 n, \qquad \textbf{(94.)}$$

in which $D =$ the corrected distance; $c =$ the constant; $a\,k =$ the stadia distance, and $n =$ the vertical angle.

Assume, as before, a vertical angle of + 10° 26′ and an intercepted distance of 3.96 feet. As each foot of the rod intercepted by the stadia wires corresponds to a distance of

100 feet, an interception of 3.96 feet corresponds to a distance of 396 feet, called herein the stadia distance, i. e., the distance from the rod to the point outside the telescope where the stadia measurement begins.

Applying the formula, we have,

$$D = 1.25 \cos 10° 26' + 396 \cos^2 10° 26' =$$
$$125 \times .98347 + 396 \times .98347^2 = 384.24 \text{ ft.}$$

To obtain the difference of level E, apply the following formula:

$$E = c \sin n + a\,k \frac{\sin 2\,n}{2}. \qquad \textbf{(95.)}$$

Applying this formula to the preceding example, we have $E = 1.25 \times .18109 + 396 \times .17810 = 70.75$, since $2\,n = 10° 26' \times 2 = 20° 52'$ and $\frac{\sin 20° 52'}{2} = \frac{.35619}{2} = .17810$.

SURVEY OF BEAVER CREEK.

Station.	Station.	Dist.	Cor. Dist.	Bearing.	Vert. Angle.	Diff. Level.	Elevation above Tide.
0	1	396	384	N 1° 15 W	+ 10° 26′	+ 70.71	1142.21
1	*A*	201		Due E	+ 20° 11′		
1	*B*	404		S 80° 10 W	− 11° 14′		
1	*C*	187		S 76° 20 W	− 14° 22′		
1	*D*	563		S 68° 32 W	+ 3° 12′		
1	2	384		N 20° 15 W	− 0° 16′		

The tables of Horizontal Distances and Differences of Elevation for Stadia Measurements are computed for observations taken on a vertical rod held perfectly plumb.

Fig. 303 shows the method of keeping sketch and notes in topographical work.

1302. An efficient topographical survey is one which fully serves every purpose for which it is made. Its value depends more upon the accuracy of that which is represented rather than the minuteness or quantity of detail. The topographer should be able to readily and intelligently decide between what is important and what is not important, and invest his time and labor accordingly, taking nothing for granted and never indulging in guesswork.

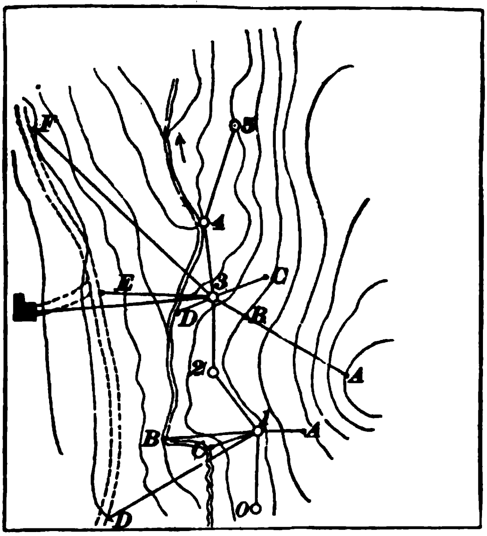


FIG. 303.

1303. The Aneroid Barometer.—Fig. 304 shows an aneroid barometer, a substitute for the mercurial barometer, which latter is not readily portable. It consists of a box of thin corrugated copper, exhausted of air. An increase in the weight of the atmosphere compresses the box, and a reduction in weight admits of the box being expanded by a spring inside. This spring is connected, by a system of levers, with a dial which indicates the pressure of the atmosphere. The face is graduated to correspond with the heights of the mercurial barometer. A thermometer is also attached to the face and shows the temperature when the readings are taken.

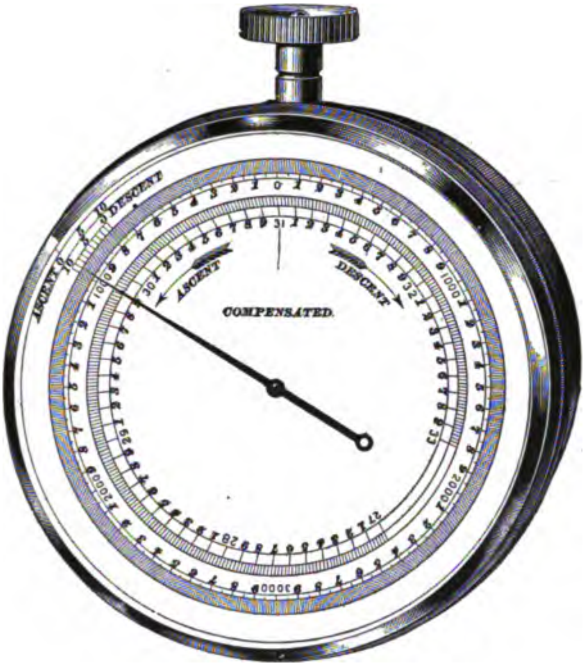


FIG. 304.

1304. How to Determine Difference in Elevations With the Aneroid Barometer.—The formula given is that used by the Engineer Corps of the United States Army. The aneroid barometers used are adjusted to agree with the mercurial barometer at a temperature of 32° Fahrenheit at the sea level in latitude 45°. Observations at the two stations whose difference in elevation is required should be made as nearly simultaneous as possible, as temperature and atmospheric conditions are constantly changing.

Let Z = difference of elevation of the two stations in feet;

h = the reading in inches of the barometer at the lower station;

H = the reading in inches of the barometer at the higher station;

t and t' = temperature (Fahr.) of the air at the two stations.

Then,

$$Z = (\log h - \log H) \times 60{,}384.3 \times \left(1 + \frac{t + t' - 64^\circ}{900}\right). \quad \textbf{(96.)}$$

EXAMPLE.—Reading at lower station, $h = 29.52$ in., $t = 70^\circ$; at higher station, $H = 27.15$ in., $t' = 62^\circ$.

$$\begin{aligned} \text{Log of } h,\ 29.52 &= 1.47012 \\ \text{Log of } H,\ 27.15 &= \underline{1.43377} \\ \text{Difference} &= .03635 \end{aligned}$$

$$1 + \frac{t + t' - 64}{900} = 1 + \frac{70 + 62 - 64}{900} = 1.0755.$$

Hence, $Z = .03635 \times 60{,}384.3 \times 1.0755 = 2{,}360.4$ feet, the difference between the elevations of the two stations.

Tables are prepared giving values of $(\log h - \log H) \times 60{,}384.3$ and of $1 + \frac{t + t' - 64^\circ}{900}$, which greatly simplifies the work of determining differences of elevations.

HYDROGRAPHIC SURVEYING.

1305. Hydrographic surveying is the process of determining, by means of soundings, the location of the deep and shallow places of harbors, sounds, rivers, etc., and recording them in charts for the use of engineers and navigators.

1306. Sounding.—Sounding is measuring the depth of water. The surface of the water forms the datum line, and the various depths measure the undulations or changes of elevation of the bottom of the body of water being sounded. The extent of knowledge of the bottom gained will depend upon the number and accuracy of the soundings.

For depths to 18 feet, a sounding rod graduated to feet and tenths is used; for greater depths, a lead line, marked to fathoms and half fathoms, is employed. It will be found necessary to keep the lead line well stretched and its length frequently tested.

1307. Conduct of Survey.—The mode of conducting a hydrographic survey is as follows: Stations at conspicuous points on shore are first carefully located by trigonometrical surveying. They form the base line by which all irregularities of shore line and the location of all soundings are determined. A good station mark is a post set firmly in the ground with about one foot of its length exposed. A hole is bored in the center of the top of the post and a flagpole set in it. The pole can be pulled out and a transit set directly over the station. Each station should be distinguished by the combination of colors on the flag, and the number of the station should be distinctly marked on the post. A permanent bench mark must be established and the height of water at the time of the soundings recorded.

Buoys are made of light wood, and painted in such colors as will make them conspicuous.

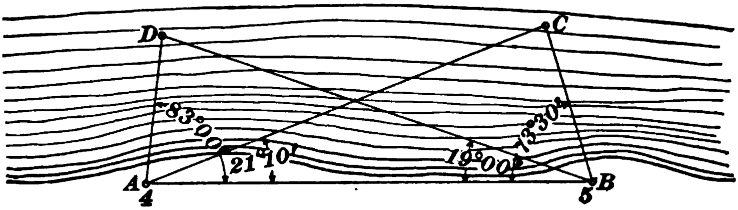


FIG. 305.

The location of buoys and soundings is illustrated in Fig. 305. The stations *A* and *B* are located and their distance apart known. A transit is set up at each station and backsighted to a rod at the other; the vernier plate is then unclamped and the leadsman in the boat is carefully followed with the instrument. At a given signal, the leadsman takes a sounding, and both instruments sight to him and read the angles, which give a side and two adjacent angles of a triangle from which to determine the location of the point *D*. In the same manner *C* and any number of points can be located. A man in the boat records the time and the soundings as they are read by the leadsman.

1308. Tide Gauges.—By means of a tide gauge the height of water at any time may be known. The datum or zero line is mean low spring tide. A simple form of tide gauge is a board nailed to the upright front of a dock. The face should be painted white and graduated to half-feet or to feet and tenths, and the zero line set at mean low spring tide. The feet marks should be in heavy black figures, so that they may be easily read.

The tide gauges used by the government are automatic, and are provided with an indicator which registers on paper the fluctuations of the tide.

LAND SURVEYING.

1309. The United States System of Surveying Public Lands.—The public lands of the United States are divided and laid out into approximately equal squares, the sides of which are true north and south or east and west lines. This is effected by means of meridian lines and parallels of latitude established six miles apart. The squares thus formed are called **townships,** and contain 36 square miles or **sections.** Each section contains, as nearly as may be, 640 acres, giving an approximate area of 23,040 acres for each township.

1310. Principal Meridians.—A **principal meridian** running due north and south and a **base line** running due east and west are established astronomically, and the half-mile, mile, and six-mile corners are permanently marked on them. These two lines form the basis of all subsequent divisions into townships and sections. All other lines, with the exception of these two and the standard parallels, are run with the compass and chain.

Fig. 306 represents a section of country thus laid out. The scale is 24 miles to 1 inch = 1520640 : 1. The diagram shows the principal meridian running truly north and south, and a base line which is a **parallel of latitude** running truly east and west. Parallel to these are other lines 6 miles apart, forming townships. All the townships situated north or south of each other form a **range,** the ranges being named by their number east or west of the principal meridian. The seven ranges east and seven west of the principal meridian, shown in Fig. 306, are described as R. 1 E, R. 1 W, etc. The townships in each range are designated by their number north or south of the base line. Thus, in the diagram,

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the township marked *A* is denoted by T. 3 N, R. 4 W; that marked *B*, by T. 2 S, R. 3 E. These abbreviations should

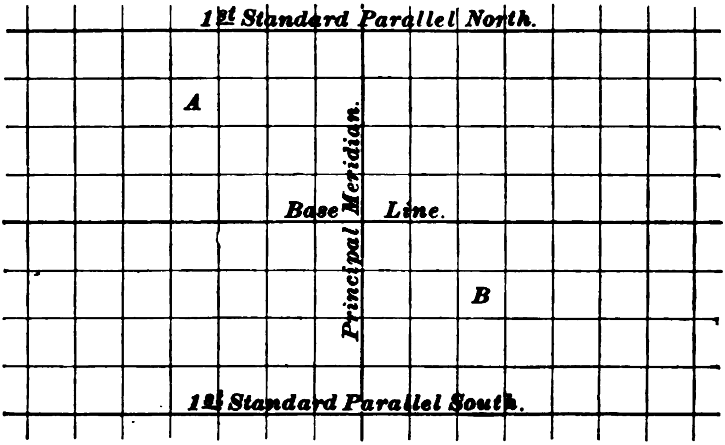
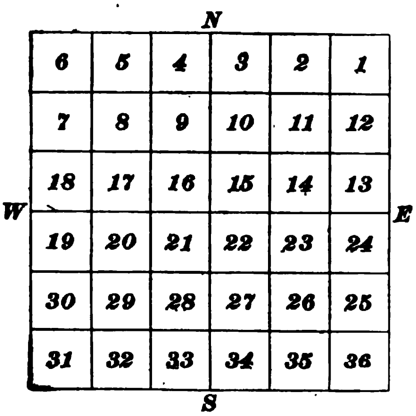


FIG. 306.

be read *township 3 north, range 4 west, and township 2 south, range 3 east.*

1311. Township Divisions.—Each township is divided into 36 sections, each one mile square and containing 640 acres as nearly as may be. The sections in each township are numbered from *1* to *36*, as shown in Fig. 307. The numbering of the sections begins at the northeast corner of the township and goes west from *1* to *6*, then east from *7* to *12*, and so on alternately until section *36* in the southeast angle of the township is reached. The sections are subdivided into quarter-sections, each half a mile square and



6	5	4	3	2	1
7	8	9	10	11	12
18	17	16	15	14	13
19	20	21	22	23	24
30	29	28	27	26	25
31	32	33	34	35	36

FIG. 307.

containing 160 acres, and sometimes into half quarter-sections of 80 acres and quarter quarter-sections of 40 acres. By this system the smallest subdivision of land can be accurately located; as, for example, the southeast quarter of section *36* in township one south in range two west of Willamette meridian.

1312. Obstacles.—The law requires that the lines of the public surveys shall be governed by the true meridian, and that the townships shall be six miles square, two conditions involving a mathematical impossibility, for strictly conforming to the meridian would necessarily throw the township out of square, for the reason that a degree of longitude, which, at the equator, is 69⅛ miles, constantly diminishes as one approaches the poles. As the meridian lines are strictly adhered to, the requirements of the law respecting areas are not fulfilled. The townships assume a trapezoidal form which increases the higher the latitude of the surveys. To meet these conditions the law provides that the sections shall contain 640 acres, as nearly as may be, and further provides that "in all cases where the exterior lines of the township thus to be subdivided into sections and half-sections shall exceed or shall not exceed six miles, the excess or deficiency shall be specially noted and added or deducted from the *western* or *northern* ranges of sections or half-sections in such township, according as the error may be in running the lines from east to west or from south to north." In order to throw the excesses or deficiencies, as the case may be, on the *north* and *west* sides of a township, according to law, it is necessary to survey the section lines from south to north on a true meridian, leaving the result in the northern line of the township to be governed by the convexity of the earth and the convergency of meridians.

Thus, suppose the land to be surveyed lies between 46° and 47° of north latitude. The length of a degree of longitude in latitude 46° north is taken as 48.0705 statute miles, and in latitude 47° north as 47.1944 miles. The difference, or convergency per square degree, = .8761 mile =

70.08 chains. The convergency per range (8 per degree of longitude) equals one-eighth of this distance or 8.76 chains, and per township (11½ per degree of latitude) will equal 8.76 chains divided by 11½ = .76 chain. Hence, we know that the width of townships along their northern boundary is 76 links less than on their southern boundary. The townships north of the base line, therefore, become narrower and narrower than the six-mile width with which they start by that amount.

1313. Standard Parallels. — Standard parallels, called **correction lines,** are established at intervals of 24 miles to provide for the correction of the error arising from the convergency of the meridians. They also serve to limit errors resulting from inaccuracies in measurement. Such correction lines when lying north of the principal base line form new base lines for the surveys north of them. The convergency or divergency is taken upon these correction lines, from which, as base lines, the townships start again with their proper widths. On these correction lines, therefore, *double corners* will be found, one set being the *closing corners* of the surveys *ending there*, and the other set the *standard corners* of the surveys *starting there.*

1314. Running Township Lines.—The principal meridian, the base line, and the standard parallels having been first astronomically run, measured, and marked according to instructions on true meridians and true parallels of latitude, the process of running, measuring, and marking the exterior lines of townships is as follows:

For townships *north* of the *base line* and *west* of the *principal meridian*, commence at Station No. *1* (Fig. 308), the southwest corner of T. 1 N, R. 1 W, as established on the base line, thence run *north* on a true meridian line 480 chains, establishing the half-mile and mile corners thereupon, according to instructions, to No. *2*, which is the northwest corner of the same township. There establish the corner of townships 1 and 2 N, ranges 1 and 2 W; thence run *east* on a *random* or trial line, setting temporary stakes at the half-mile and mile points and noting the distance where the line intersects the eastern boundary north or south of the *true* or established corner. Run and measure *westward* on the *true* line, establishing permanent half-mile and mile corners, noting all water crossings and the character of the land, as per instructions, to No. *4*, which is identical with No. *2*. The last half-mile will fall short of 40 chains by about the amount of the calculated convergency per township, which, in the above supposed case, equals 76 links.

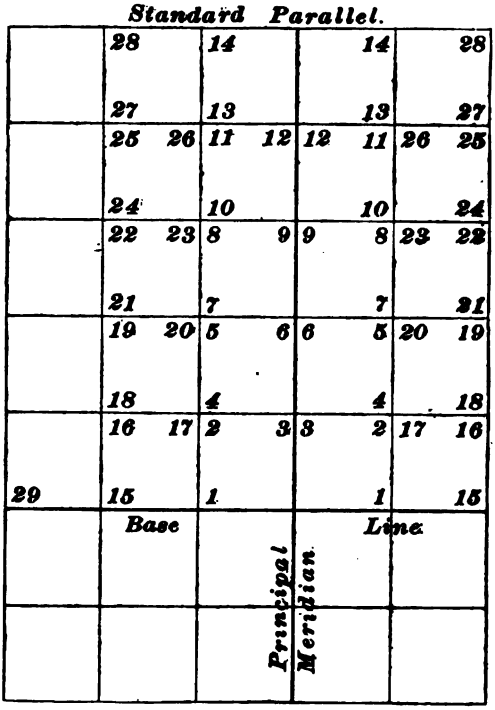


FIG. 308.

The terms random and true lines are explained in Fig. 309. The boundary from Station *1* to Station *2* is a true meridian and 480 chains in length, with *permanent* corners set at each half-mile and mile. From *A* run and measure towards B on a true east and west line, as shown by the dotted line $A\ B$, which is called a **random line,** setting temporary half-mile and mile posts. This random or trial line being run on a parallel of latitude, must intersect the principal meridian near the true corner C, previously established. The return or true line always connects this true corner with the one from which the random starts. Random lines

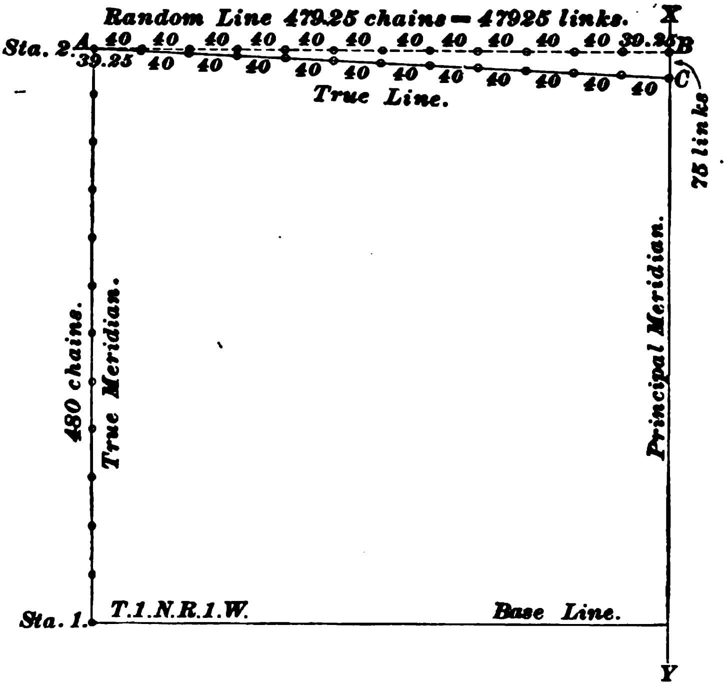


FIG. 309.

are either true east and west or true north and south lines, i. e., they are either parallels of latitude or true meridians. Suppose the random line intersects the principal meridian $X\ Y$ at B, 75 links to the north of the true or established corner at C, the length of $A\ B$ is 479 chains and 25 links. The triangle $A\ B\ C$ is a right-angled triangle, right angled at B; dividing $B\ C = 75$ links by $A\ B = 47925$ links, we have the tangent of the angle $B\ A\ C = \frac{75}{47925} = .00156 =$

tan 0° 05′. The angle $B\ C\ A$ is, therefore, $90° - 0° 05' = 89° 55'$, and the return course, or true line, $C\ A$ is N 89° 55′ W. Setting the instrument over the true or established corner C, the compass is set for the true course $C\ A$, N 89° 55′ W, and measuring 40 chains from C, a permanent half-mile or quarter-section post is set, 40 chains further a mile or section post is set, and so on, setting half-mile and mile posts at regular intervals of 40 chains until the last half-mile post is set; between it and the township corner A, the distance is but 39 chains and 25 links, thus leaving the deficiency in the western tier of sections as prescribed by law.

In case the random line materially falls short, or overruns in measurement, or intersects the eastern boundary at a considerable distance from the established corner, it will be evident that there has been considerable error either in direction or measurement of the lines, or both, and the lines must be *retraced* even if it should be found necessary to rerun the meridianal boundaries of the township (especially the western boundary) so as to discover and correct the error. The true corners must be established, and the false ones destroyed and obliterated, and all facts carefully set forth in the notes so as to avoid future confusion.

Then proceed north from *4* to *5*, establishing corners as before; No. *5* is the N W corner of T. 2 N, R. 1 W; east to No. *6* (the N E corner of the same township), west to No. *7* (the same as No. *5*), north to No. *8* (the N W corner of T. 3 N, R. 1 W), east to No. *9* (the N E corner of the same township), west to No. *10* (same as No. *8*), north to No. *11* (the N W corner of T. 4 N, R. 1 W), east to No. *12* (the N E corner of the same township), west to No. *13* (same as No. *11*), and thence north on a true meridian to the standard parallel or correction line (which is here five townships, or 30 miles, north of the base line), throwing the difference over or under four hundred and eighty chains on the last half mile, according to law. At the intersection with the standard parallel establish the *closing corner*, the distance of which from the standard corner must be measured and noted as required by the instructions. In case any obstruction should have prevented the extension of the standard parallel along the field of the present survey, the surveyor will establish a corner for the township, subject to correction, should the parallel be extended. The surveyor then returns to the base line, and, from the southwest corner of T. 1 N, R. 2 W, carries up another tier of townships, closing as before.

For townships *north* of the base line and *east* of the principal meridian the order of survey is as follows: Beginning at the southeast corner of T. 1 N, R. 1 E, proceed as with townships *north* and *west*, except that the trial or random line is run and measured *west* and the **true line** *east*, throwing the difference over or under 480 chains on the *west* end of the line. Accordingly, the surveyor, having measured his trial line west, will first determine the length of the last half-section line, and commence the measurement of the true line with such excess or deficiency, and, consequently, the remaining measurements will all be exact half miles and miles.

1315. Running Section Lines.

—The interior or sectional lines of all townships, however situated with reference to base and meridian lines, are laid off and surveyed, as shown in Fig. 310.

In this figure the squares and large figures represent sections; the small figures are referred to in the following directions. Commence at No. *1* (see small figure in the diagram) which is a township boundary for sections *1*, *2*, *35*, and *36*; thence run *north* on a true meridian; at 40 chains establish a half-mile or quarter-section post, and at 80 chains establish the corner of sections *25*, *26*, *35*, and *36*. Thence *east* on a *random line* to No. *3*, setting a temporary quarter-section post at 40 chains, noting the measurement to No. *3* and the distance of the random's intersection north or south of the true or established corner of sections *25*, *36*, *30*, and *31*. Thence *correct west* on a *true line* to No. *4*, setting the quarter-section post on this line *equidistant* from the two corners whose distance apart is now known. In like manner proceed from *4* to *5*, *5* to *6*, *6* to *7*, and so on to No. *16*, the corner of sections *1*, *2*, *11*, and *12*, thence *north* on a *random line* to No. *17*, setting a temporary quarter-section post at 40 chains and noting the length of the whole line and the distance of the *random's* intersection east or west of the true corner of sections *1*, *2*, *35*, and *36* established on the township boundary, then *southwardly* from the latter on a true line, noting the *course* and distance to No. *16*, the established corner to sections *1*, *2*, *11*, and *12*, care being taken to establish the quarter-section post at 40 chains from said section corner, thus throwing the excess or deficiency on the northern half mile, according to law. Proceed in like manner through all the intervening tiers of sections to No. *73*, the corner of sections *31*, *32*, *5*, and *6*. Thence north on a true meridian 80 chains to *74*, setting a quarter-section post at

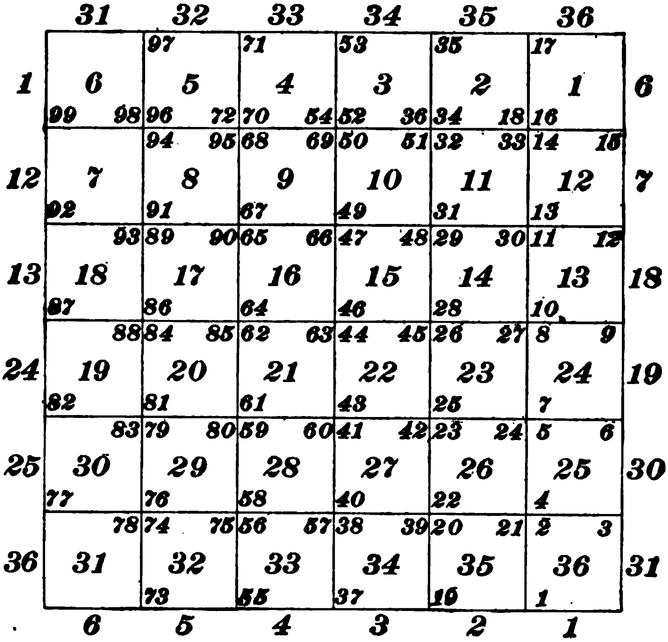


FIG. 310.

40 chains, and at 80 chains setting corner of sections *29*, *30*, *31*, and *32*; then east on a random to *75*, setting temporary quarter-section post at 40 chains, noting the entire measurement to the eastern boundary and the distance of the random's intersection north or south of the true corner of sections *28*, *29*, *32*, and *33*; thence west on a true line, setting the quarter-section post on the true line and equidistant from either end, to No. *76*, which is identical with *74*; thence west on a random line to *77*, setting temporary quarter-section post at 40 chains, noting the full measurement of the line and the distance of the random's intersection with the township boundary *north* or *south* of the established corner of sections *30*, *31*, *25*, and *36*; thence eastwardly on the true line, *giving its course* and setting the quarter-section post 40 chains from the corner of sections *29*, *30*, *31*, and *32*, thus throwing the excess or deficiency of measurement on the western half mile of the section according to law. Proceed *north* in like manner from No. *78* to *79*, *79* to *80*, *80* to *81*, and so on to No. *94*, the southeast corner of section *6*, where, having established the corner of sections *5*, *6*, *7*, and *8*, run thence successively the random line *east* to *95*, *north* to *97*, and *west* to *99*, and by reverse courses *back on true lines* to the southeast corner of section *6*, establishing the quarter-section corners, and noting *courses*, measurements, and distances as prescribed by law.

In townships contiguous to standard parallels the above method is varied as follows: In every township *south* of the principal *base line* which *closes* on a standard parallel, the surveyor will begin at the southeast corner of the township and measure westward, establishing the half-mile and mile corners and noting their distance from the preestablished corners. He will then proceed to subdivide as directed under the above head.

In townships *north* of the *principal base line* which *close* on the standard parallel, the section lines must be closed on the standard parallel with true *meridian lines* instead of *course* lines, as directed for townships otherwise situated; and the connections of the closing corners with the preestablished standard corners are to be ascertained and noted.

In case the surveyor is unable to close the lines on account of the standard not having been run for some reason, as before mentioned, he will then plant a temporary post or construct a mound at the end of the *sixth* mile, thus leaving the lines and their connections to be finished when the standard shall have been run.

1316. Water Frontage.—Departures from the general system of dividing land have been authorized by law, especially in the case of water frontage.

In surveying the public lands of Louisiana, which border on rivers, streams, lakes, and bayous, surveyors were authorized to divide the land with water frontages of fifty-eight poles and running back four hundred and sixty-five poles in depth, "and of such shape and bounded by such lines as the nature of the country will render practicable and most convenient." Later, authority was given to survey lands with two acres water frontage and running back a depth of forty acres, tracts so surveyed to be offered for sale entire instead of in half quarter-sections. In localities where it would best subserve the interests of the people to have fronts on the navigable streams and running back into the uplands for timber, surveyors were authorized to increase the quantity of land so as to give four acres frontage and forty acres in depth, giving tracts of 160 acres, but in so doing they were only to survey the lines between *every four lots* (or 640 acres), establishing the boundary posts or mounds *in front* and *in rear*, at the distances requisite to secure the quantity of 160 acres to each lot, either rectangularly where practicable or at oblique angles where otherwise. The angle is not important so long as the principle is adhered to of making, as far as possible, the rear lines square with the regular sectioning.

1317. Meandering.—This name is applied to the usual mode of traversing or surveying a navigable stream. The instructions for this work are in part as follows: Both banks of *navigable* rivers are to be meandered by taking the courses and distances of their sinuosities and the same are to be entered in the *meander* field book. At those points where either the township or section lines intersect the banks of a navigable stream, posts, or, where necessary, *mounds of earth or stone* are to be established at the time of running these lines. These are called "meander corners," and, in meandering, the surveyor will commence at one of these corners on the township line, coursing the banks and measuring the distance of each course from the commencing corner to the next "meander corner" upon the same or another boundary of the same township, carefully noting intersections with all the intermediate meander corners. By the same method meander the opposite banks of the river.

The crossing distance *between* the *meander corners* on the same line is to be ascertained by triangulation, in order that the river may be protracted with entire accuracy. The particulars are to be given in the field notes. The courses and distances on meandered navigable streams are the bases for the calculation of the true areas of the tracts of land (sections, quarter-sections, etc.), known to the law as *fractional* and bounding on such streams.

The surveyor is also to meander, in manner aforesaid, all lakes and deep ponds of the area of twenty-five acres and upwards, also navigable bayous.

As traverse tables are generally calculated to 15′ angles, it is advisable to make meander courses read to quarter degrees instead of intermediate minutes, except in closing or where the extreme length of a side of a lake or stream falls in one course.

The precise relative position of islands in a township made fractional by the river in which they are situated is to be determined trigonometrically. To meander islands crossed by government lines, meander corners are previously established at opposite points on the shore of the island, and the meanders run from one to the other. Should the island not be crossed by a line, measure a special base line from the meander corner nearest to the island, triangulating to and establishing at any convenient point on the island a special meander corner from and to which the meanders of the island start and close.

1318. Marking Lines.—All lines on which are to be established the legal corner boundaries are to be marked after this method, viz.: Those trees which may intercept the line must have two chops or notches cut on each side of them without any other marks whatever; these are called *sight trees* or *line trees.* A sufficient number of other trees standing nearest to the line on either side of it are to be *blazed* on two sides diagonally or quartering towards the line, in order to render the line conspicuous and readily traced, the blazes to be opposite to each other, coinciding in direction with the line where the trees stand very near it, and to approach nearer each other the further the line passes from the blazed trees. Due care must ever be taken to have the line so well marked as to be readily followed.

1319. Marking Corners.—After a true coursing and most exact measurements, the corner boundary is the consummation of the work for which all the previous pains and expenditure have been incurred. A boundary corner in a timbered country is to be a *tree*, if one be found at the precise spot; and if not, a *post* is to be planted thereat, and the position of the corner post is to be indicated by trees adjacent (called bearing trees), the angular bearings and distances of which from the corner are facts to be ascertained and recorded by the surveyor. In a region where stones abound, the corner boundary will be a small monument of stones alongside of a single marked stone for a township corner and a single stone for all other corners.

In a region where neither timber nor stone is available, the corner will be a mound of earth of prescribed size varying to suit the case.

When *posts* are used, their length and size must be proportional to the importance of the corner, whether township, section, or quarter-section post.

Township corner posts are three inches square and set at least twenty-four inches above ground.

Where a township post is at a corner, common to *four* townships, it is to be set in the ground diagonally, as shown in Fig. 311, and the cardinal points of the compass indicated by lines cut or sawed out of its top at least one-eighth of an inch deep, as shown in the figure. On each face of the post is to be marked the number and range of the particular township which it *faces*. Thus, if the post be a common boundary to four townships, viz., one and two south of the base line and range two west, and also one and two south of the base line and range three west, the face markings will be as follows:

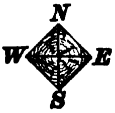


Fig. 311.

From N to E	R. 2 W T. 1 S S 31
From N to W	3 W 1 S 36
From E to S	2 W 2 S 6
From W to S	3 W 2 S 1

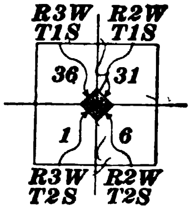


Fig. 312.

The position of the post which is here taken as an example is shown in Fig. 312.

These marks are neatly chiseled into the wood, and are also marked with *red chalk.* The *number* of the *sections* which they respectively *face* will also be marked on the township post.

Section or mile posts, being corners of sections, when they are common to four sections, are to be set *diagonally* in the earth (in the manner provided for township posts), and with similar marks cut in the top to indicate the cardinal points of the compass, while on *each side* of the post is cut the number of the *particular section* which the side faces. Also, on one side is to be marked the number of its township and range. To make such marks more conspicuous and durable, red chalk is applied. A quarter-section or half-mile post is to have no other mark than ¼ S to indicate what it stands for.

Township posts are to be *notched* with *six* notches on each edge or angle corresponding to the cardinal points of the compass. All mile posts on township lines must have as many notches on opposite angles as they are miles distant from the corresponding township corners. Each of the posts at the corners of sections in the interior of a township must have on their four angles, corresponding to the cardinal points, as many notches as they are miles distant from the corresponding township corners. The four sides of the post will indicate the numbers of the sections which they respectively face. Should a tree be found at the place of any corner it will be marked and notched in the manner before described and will serve in place of a post; the kind of tree and the diameter must be given in the field notes.

The position of all corner posts or corner trees of whatever description, which may be established, is to be perpetuated in the following manner, viz.: From such post or tree, the courses shall be taken and the distances measured to two or more adjacent trees in opposite directions as nearly as may be, which are called *bearing trees*, and are to be blazed near the ground with a large blaze facing the post and having one notch in it, neatly and plainly made with an ax, square across, and a little below the middle of the blaze. The kind of tree and the diameter of each are facts to be clearly set forth in the field book.

On each bearing tree the letters B. T. must be distinctly cut into the wood in the blaze a little above the notch or on the bark, with the number of the range, township, and section.

At all township corners and at all section corners on range or township lines *four* bearing trees are to be marked in this manner, one in each of the adjoining sections.

At interior section corners *four* trees, one to stand within *each* of the four sections to which such corner is common, are to be marked in the manner aforesaid if such be found.

From quarter-section and meander corners, two bearing trees are to be marked, one within each of the adjoining sections. Stones at township corners (a small monument of stones being alongside thereof) must have *six* notches cut with a pick or chisel on each edge or side towards the cardinal points; and where used as corners in the interior of a township, they will also be notched with a pick or chisel to correspond with the directions given for notching posts similarly situated.

Stones when used as quarter-section corners will have ¼ cut on them, on the *west* side in *north* and *south* lines, and on the *north* side in *east* and *west* lines.

Wherever bearing trees are not found, *mounds* of earth or stone are to be raised *around posts* on which the corners are to be marked in the manner aforesaid. Wherever a mound of earth is adopted, the same will present a pyramidal shape. At its base on the earth's surface a *quadrangular trench* will be dug; a spade deep of earth being thrown up from the sides of the line *outside* the trench, so as to form a *continuous elevation along its outer edge.* In mounds of earth common to four townships or four sections, they will present the angles of the quadrangular trench *diagonally* to the cardinal points. In mounds common only to *two* townships or *two* sections, the sides of the trench will face the cardinal points. Prior to piling up the earth, in a *cavity*, formed at the corner boundary point, is to be deposited a *stone*, or a portion of charcoal; or a charred stake is to be driven twelve inches down into such center point to be a witness for the future. The surveyor is further specially enjoined to plant midway between each pit and the trench seeds of some tree, those of fruit trees adapted to the climate being always to be preferred.

Double corners are to be found nowhere except on the standard parallels or correction lines whereon are to appear both the corners which mark the intersection of the lines which close thereon and those from which the surveys start in the opposite direction.

The corners which are established on the standard parallel at the time of running it are to be known as "*standard corners*," and in addition to all the ordinary marks (before described) they will be marked with the letters S. C. The *closing corners* will be marked C. C.

1320. Field Books.—There are several field books, viz.:

1. Field Books for the meridian and base lines, showing the establishment of *township*, *section*, or mile, and *quarter-section*, or half-mile boundary corners thereon; with the crossings of streams, ravines, hills, and mountains; the character of the soil, timber, minerals, etc. These notes will be arranged in series by *mile stations* consecutively from number one to number ——.

2. Field Books for the *standard parallels* or correction lines, showing the establishment of the township, section, and quarter-section corners, besides exhibiting the topography of the country on line as required on the base and meridian lines.

3. Field Books for exterior lines of townships, showing the establishment of the corners on line, and the topography as aforesaid.

4. Field Books for the subdivision of townships into sections and quarter-sections; at the close whereof will follow the notes of the meanders of navigable streams. Those notes will also show by ocular observation the estimated rise and fall on the line. A description of the timber, undergrowth, surface soil, and minerals upon each section line is to follow the notes thereof, and not be intermixed with them.

5. The Geodetic Field Book, comprising all triangulations, angles of elevation and depression, leveling, etc.

1321. Retracing Old Lines.—The original surveys of lands in the older States of the American Union were imperfectly made and full of errors. This was owing to two principal causes; viz., the cheapness of the lands and the lack of skill in the surveyors. Boundary lines described in deeds and shown in maps as straight are found to be crooked on the ground; tracts contain less or more land than called for in descriptions. Records of adjoining tracts make one to overlap another or leave an unclaimed gore between them. These discrepancies and blunders often render the work of the surveyor, when retracing old boundaries or establishing corners, exceedingly difficult, and great tact and judgment are often necessary in making amicable and satisfactory adjustments of contending claims. In general, old boundaries, such as line trees, stone monuments, and fences are accepted as holding ; but, before retracing lines the surveyor should, if possible, secure the consent of adjacent owners to abide by such monuments and boundaries, irrespective of the lines or quantities called for in contracts or deeds. It must be borne in mind that the bearings of lines are each year undergoing a slight change which, in a long period, amounts to several degrees, and if the lines were rerun according to original bearings as given in descriptions, they would enclose a tract differing widely from that included in the original survey. The surveyor must accord-

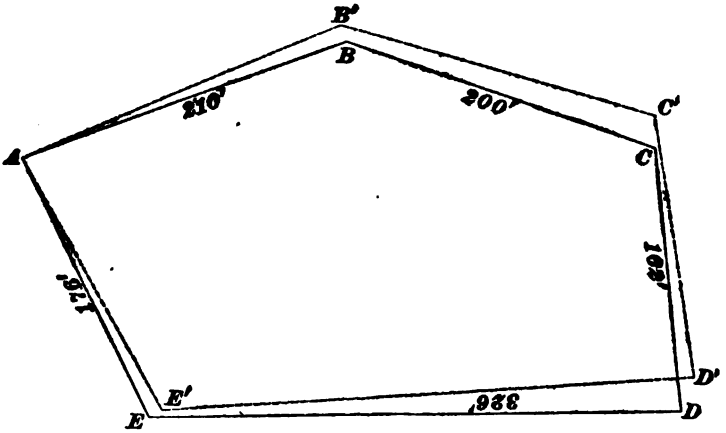


FIG. 313.

ingly determine the amount of magnetic variation or change which has taken place between the time of the original survey and the date of the survey about to be made, and having determined such change or variation, he must make the original bearings conform to the calculated variation before commencing the survey.

Fig. 313 illustrates the effect of magnetic variation in altering the direction of lines. The figure *A B C D E* gives the outline of a tract according to the original survey, and *A B' C' D' E'* the relative directions of the boundaries when resurveyed with the original bearings, there having been during the intervening time a change in magnetic variation of 3° west.

Let columns 1, 2, and 3 in the accompanying diagram give the courses, original bearings, and distances, and column 4 the corrected bearings which the original boundaries will have, when allowance has been made for the magnetic variation. When the north end of the needle has been moving westerly, i. e., when the variation or *change* is *west*, the *corrected* or *present bearings* will be the *sums* of the change and the old bearings which were *northeasterly* or *southwesterly* and the *differences* of the change and the old bearings which were *northwesterly* or *southeasterly;* when the variation or change is *easterly*, the corrected or present bearings will be the *differences* of the change and the old bearings which were *northeasterly* or *southwesterly* and the *sums* of the change and the old bearings which were *northwesterly* or *southeasterly*.

1 Courses.	2 Bearings.	3 Distances.	4 Corrected Bearings.
A B	N 68° 00' E	210	N 71° 00' E
B C	S 73° 00' E	200	S 70° 00' E
C D	S 8° 00' E	162	S 5° 00' E
D E	S 87° 00' W	326	S 90° 00' W
E A	N 28° 00' W	176	N 25° 00' W

It will be seen, by reference to Arts. **1211** and **1212,** that *declination* is the reverse of *variation*, i. e., a *west* declination results when the variation or movement of the N end of the needle is to the east, and *east* declination results when the movement of the N end of the needle is to the west. By this rule the bearings given in column 4 are obtained. Before commencing the survey, the surveyor should correct all the bearings and write them out together with the original bearings in their proper order.

1322. How to Determine Magnetic Variation.—If the date of the original survey is known, the amount of variation may be determined from published tables giving the yearly variation for different sections of the country, but the date of the survey is often omitted. The date of the deed *must not* be taken as the date of the survey.

If one of the original boundaries remains unchanged, the magnetic variation can be determined at once by taking the present bearing of the line. The difference between the present bearing and that of the original survey is the required correction. The corrections are then to be made in the original bearings and the resulting courses run out. Where the measurements fall short of or overrun the original measurements, corrections must be made, locating the original corners if they can be found or establishing new ones, and, if possible, to the mutual satisfaction of adjoining proprietors.

1323. Establishing New Boundaries.—Where the description and map show a boundary to be a straight line and the actual boundary is found to be crooked, it is a good policy to establish a new and straight boundary by the principle of "give and take," providing adjoining owners will agree to the adjustment.

Fig. 314 illustrates the principle which is frequently employed in correcting such boundaries.

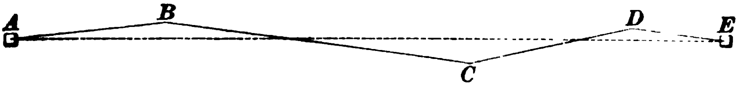


FIG. 314.

Let *A* and *E* be two corners and let the boundary line joining them be described and shown in the map as a straight line. Let the irregular line *A B C D E* represent the actual boundary. It is evident that the dotted straight line *A E* may be substituted for the irregular line *A B C D E*, and would equitably divide the adjoining properties. The principle of give and take is applied, the adjoining owners making exchanges of equal areas.

The location of the new boundary is determined by making a careful survey of the old boundary and platting it to a large scale; a fine thread is then stretched on the plat and a line of division made as closely as may be estimated by the eye. The areas of the equalizing triangles are then calculated by scaling their dimensions, and if they do not balance the dividing line can readily be shifted until the desired result is obtained. The line is then measured on the ground and permanent corners established. Where the boundary is in woodland, careful search must be made for line and bearing trees. Blaze marks are very enduring, being easily recognized on some varieties of trees after a lapse of a quarter of a century.

1324. Lost and Obliterated Corners.—Corner monuments of perishable material, such as wooden posts, decay and in time become obliterated. A pile of stones, which is commonly used as a corner, may become scattered, and, unless permanent witnesses remain, it may be a difficult matter to restore the landmark. The most enduring witnesses are live trees which are disposed as shown in Fig. 315.

Three trees facing the corner are chosen ; in each tree three notches are cut in the side facing the corner, and the bearing and distance from each to the corner are recorded in the notes. A sketch is made in the note book giving the relative positions of the corner and the witness trees. When the corner is lost, but the witness trees still remain, the corner is restored by describing intersecting arcs from the witness trees as centers with radii equal to the given distances from the original corner. Where both corner and witnesses are gone, it is best to run from both directions towards the missing corner, placing the corner at the intersection of the lines. The surveyor need not expect to find his measurements agree with those in original surveys, but he can save his successor much annoyance and trouble by careful and accurate work. He should always give both in map and in description the exact date of the survey; the direction of courses should also be given both in writing and figures, and the corners should be fully described. A stone monument is the best corner, and should always be used where the material is available.

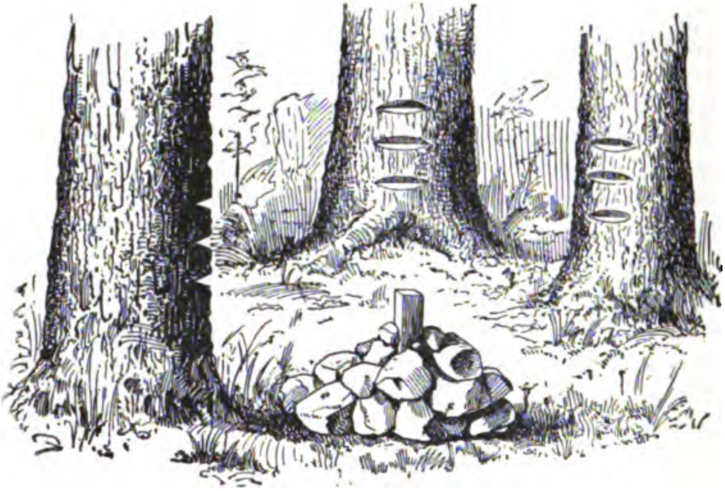


FIG. 815.

AREAS.

1325. The **area** of a surface is its superficial content. In the surveying of public lands all measurements are made with the surveyor's chain, commonly known as Gunter's chain, from the name of the inventor. It is 66 feet in length and contains 100 links, each 7.92 inches long. At each interval of ten links a brass tag is attached with tally points similar to those on the engineer's chain described in Art. **1214.** Tables of surveyor's linear and square measure are given in Arts. **209** and **211.** For land areas the unit of measurement is the square foot = 144 square inches, though areas of considerable extent are usually expressed in acres. An acre contains 43,560 square feet of surface. Rectangular areas are determined by multiplying the length in feet by the breadth in feet, and dividing the product by 43,560, which gives the area in acres.

In surveys of farms or larger tracts, dimensions are given in chains and links. The product of such dimensions is in square chains, which, divided by 10 (the number of square chains in an acre), gives the area in acres.

EXAMPLE.—A rectangular piece of land is 1,060 feet in length by 820 feet in breadth; required, the area.

SOLUTION.— 1,060 × 820 = 869,200 sq. ft. 869,200 ÷ 43,560 = 19.954 acres. Ans.

DIFFERENT METHODS OF COMPUTING AREAS.

1326. By Dividing the Plat into Triangles.—Farms, especially in the older States of the Union, are commonly of irregular form. The readiest and—where the measurements have been accurately made—a sufficiently accurate method of determining areas is as follows: Make an accurate plat of the tract to as large a scale as may be conveniently used. Divide the resulting figure, an irregular polygon, into triangles, making their sides of as nearly equal length as possible. It is evident that the sum of the areas of the several triangles into which the polygon is divided is equivalent to the area of the polygon. This mode of calculating area is illustrated in Fig. 316.

Let the irregular polygon $A\,B\,C\,D\,E\,F$ be the outline of a tract of land the area of which is required. Draw the diagonals $B\,F$, $C\,F$, and $C\,E$, dividing the figure into four triangles, the combined area of which is equal to the area of the polygon. From the vertexes A, B, D, and E drop the perpendiculars $A\,G$, $B\,H$, $D\,K$, and $E\,L$ upon the opposite bases of the triangles. The lengths of the several bases and altitudes are measured with the scale and the areas of the several triangles calculated by the rule: *the area of a triangle is equal to one-half the product of its base and altitude.*

The sum of the areas of the several triangles is equal to the area of the polygon.

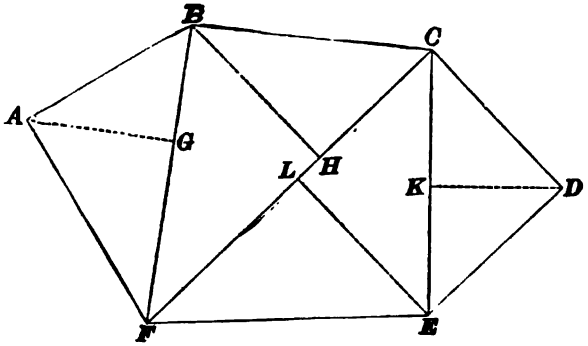


FIG. 316.

1327. By Dividing the Plat into Trapezoids.—A plat of the area having been made, it may be resolved into

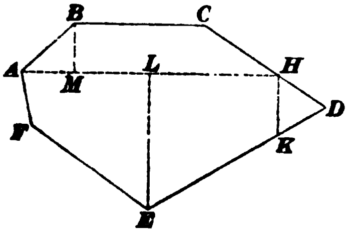


FIG. 317.

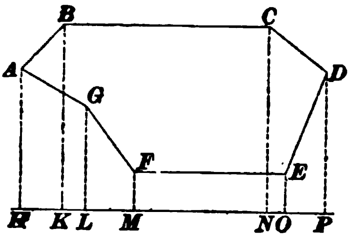


FIG. 318.

trapezoids by either of the methods shown in Figs. 317 and 318. In Fig. 317 the line *A H* is drawn parallel to *B C*, and the lines *B M*, *K H*, and *E L* are drawn perpendicular to *A H*, dividing the figure into trapezoids and the triangle *H D K*. The area of each trapezoid is equal to one-half the sum of its bases multiplied by its altitude, and the sum of their areas together with the area of the triangle is equivalent to the area of the polygon *A B C D E F*. In Fig. 318 a base line *H P* is drawn, and from each angle of the polygon perpendiculars are drawn to it. The sum of the areas of the three trapezoids *A B K H*, *B C N K*, *C D P N* is found, and from that sum, the sum of the areas of the trapezoids *A G L H*, *G F M L*, *F E O M*, and *E D P O* is subtracted. The difference of these sums is the area of the polygon *A B C D E F G.*

LATITUDES AND DEPARTURES.

1328. Definitions.—The **latitude** of a point is its distance north or south of some "*parallel of latitude*" or line running *east* and *west*. The **longitude** of a point is its distance east or west of some *meridian* or line running *north* and *south.*

The meridian from which the longitude of a point is reckoned is the *magnetic meridian.*

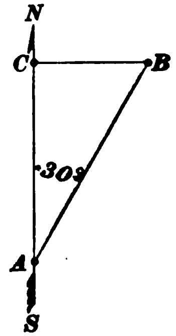


FIG. 319.

The distance which one end of a line is due north or south of the other end is called the **latitude** of that line.

The distance which one end of a line is due east or west of the other end is called the **departure** of that line.

The latitude and departure of a line and its determination are explained in Fig. 319. Let *A B* be the given line whose length and angle with the magnetic meridian *N S* is known, and whose latitude and departure are required. From *B* draw *B C* perpendicular to *N S*, forming the right-angled triangle *A C B*, in which the sides *C A* and *C B* about the right angle are, respectively, the latitude and the departure of the line *A B.* Then,

$$A\,C = A\,B \times \cos \text{ bearing, and}$$
$$B\,C = A\,B \times \sin \text{ bearing;}$$

that is, the latitude is equal to the product of the cosine of the bearing and the length of the course; and the departure is equal to the product of the sine of the bearing and the length of the course.

Let $A\,B = 400$ feet and the bearing of *A B*, i. e., the angle $B\,A\,C = 30°$. Then, latitude $A\,C = \cos 30° \times 400 =$

.86603 × 400 = 346.412 ft., and departure *B C* = sin 30° × 400 = .50000 × 400 = 200 ft.

If the course be northerly, the latitude will be north, marked +, and be additive; if southerly, it will be marked —, and be subtractive. If the course be easterly the departure will be east, marked +, and be additive; if westerly, the departure will be west, marked —, and be subtractive.

1329. Traverse Tables.—The latitude and departure of any distance for any bearing can be found by a table of natural sines and cosines, but for facilitating work special tables, called traverse tables, have been prepared. They usually give the latitude and departure for any bearing to each quarter of a degree and for distances from 1 to 9.

To *use* the tables (see traverse tables, or Latitudes and Departures of Courses), find the number of degrees in the bearing in the left-hand column if the bearing be less than 45°, and in the right-hand column if the bearing be greater than 45°. The numbers on the same line running across the page are the latitudes and departures for that bearing and for the respective distances, 1, 2, 3, 4, 5, 6, 7, 8, 9, which appear at the top and bottom of the pages, and which may be taken to represent links, rods, feet, chains, or any other unit. Thus, if the bearing be 10° and the distance 4, the latitude will be 3.939 and the departure .695; with the same bearing, and the distance 8, the latitude will be 7.878 and the departure 1.389, or double the latitude and departure for the distance 4. Any distance, however great, can have its latitude and departure readily obtained from this table, since, for the same bearing, the latitude and departure are directly proportional to the distance because of the similar triangles which they form. Hence, the latitude and departure for 80 is ten times the latitude and departure for 8, and is found by moving the decimal point one place to the right; that for 500 is 100 times the latitude and departure for 5, and is found by moving the decimal

point two places to the right, and so on. By moving the decimal point one, two, or more places to the right the latitude and departure may be found for any multiple of any number given in the table. In finding the latitude and departure for any number such as 453, the number is resolved into three numbers, viz.; 400 and the latitude and departure
50 for each taken from the table
3 and then added together.
453

We thus obtain the following

Rule.—*Write down the latitude and departure, neglecting the decimal points, for the first figure of the given distance; write under them the latitude and departure for the second figure, setting them one place further to the right; under these place the latitude and departure for the third figure, setting them one place still further to the right, and so continue until all the figures of the given distance have been used; add these latitudes and departures and point off on the right of their sums a number of decimal places equal to the number of decimal places to which the tables being used are carried; the resulting numbers will be the latitude and departure of the given distance in feet, links, chains, or whatever unit of measurement is adopted.*

Example.—A bearing is 16° and the distance 725; what is the latitude and departure?

Distances.	Latitudes.	Departures.
700	6729	1929
20	1923	0551
5	4806	1378
725	696.936	199.788

Solution.—Taking the nearest whole numbers and rejecting the decimals, we find the latitude and departure to be 697 and 200.

When a 0 occurs in the given number the next figure must be set *two* places to the right, as in the following example:

Example.—The bearing is 22° and the distance 907 feet; required, the latitude and departure.

SOLUTION.—

Distances.	Latitudes.	Departures.
900	8345	3371
7	6490	2622
907	840.990	339.722

Here the place of 0 in both the distance column and in the latitude and departure columns is occupied by a dash —. Rejecting the decimals, the latitude is 841 feet and the departure 340 feet. When the bearing is more than 45°, the names of the columns must be read from the bottom of the page. The latitude of any bearing, as 60°, is the departure of its complement, 30°; and the departure of any bearing, as 30°, is the latitude of its complement, 60°. This will be readily understood from an inspection of Fig. 320, in which if NS be the magnetic meridian and $BOA = 60°$ the bearing, then AO is the latitude and AB the departure. If, now, OC be made the meridian and $BOC = 30°$ (the complement of BOA) the bearing, then OC (the equal of AB) is the latitude, and BC (the equal of AO) the departure.

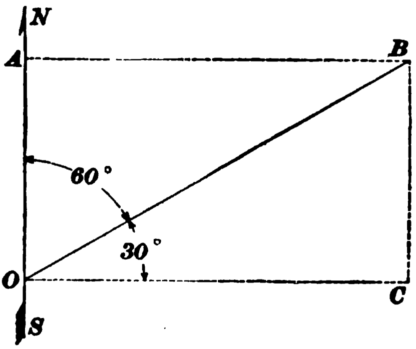


FIG. 320.

EXAMPLE.—Let $OB = 1{,}326$ feet, and its bearing $= 60°$.

SOLUTION.—

Distances.	Latitudes.	Departures.
1,000	0500	0866
300	1500	2598
20	1000	1732
6	3000	5196
1326	663.000	1148.316

The required latitude is 663 feet and the departure 1,148 feet.

Where the bearings are given in smaller fractions of degrees than is found in the table, the latitudes and departures can be found by interpolation.

Traverse tables are chiefly employed in testing the accuracy of surveys, platting them, and calculating their content.

1330. Testing a Survey.—When a surveyor has completed the survey of a field or farm by taking bearings and measuring courses, it is evident that he has gone as far north as south and as far east as west. The sum of the north latitudes shows how far north he has gone, and the sum of the south latitudes shows how far south he has gone. The sum of the east departures shows how far east he has gone, and the sum of the west departures shows how far west he has gone. Hence, if the survey has been correctly made these sums will be equal or will balance.

The entire operation of testing a survey is illustrated in the following example:

Stations.	Bearings.	Distances.	Latitudes.		Departures.	
			N +	S −	E +	W −
1	N 34¼° E	273	226		154	
2	N 85½° E	128	10		128	
3	S 56¾° E	220		121	184	
4	S 34¼° W	353		292		199
5	N 56½° W	320	177			267
			413	413	466	466

34¼°	1658	1126	85½°	0078	0997
273	5786	3940	128	0157	1994
	2480	1688		0628	7975
	225.640	153.688		9.998	127.615
56¾°	1097	1673	34¼°	2480	1688
220	1097	1673	353	4133	2814
	120.67	184.03		2480	1688
				291.810	198.628

56½°	1656	2502
320	1104	1668
	176.64	266.88

Adding up the north and south latitudes we find them to exactly balance each other, as do the east and west departures, which proves the survey to be correct. On account of the inherent defects of the compass and the errors which are liable to occur in measurement, especially on rough and extensive areas, it is but rarely that the survey will exactly balance. A moderate discrepancy, which would indicate what may be called unavoidable errors, will be allowable, and the survey accepted as correct. How great a difference in the sums of the columns may be allowed is a doubtful question. Every surveyor of experience knows the average degree of accuracy of his work, and will readily distinguish between a serious error and an allowable inaccuracy.

1331. Balancing a Survey.—When the sums of the latitudes and of the departures do not equal each other, and yet the difference does not indicate any error, the different latitudes and departures are modified so that their sums shall be equal. This process is called *balancing the survey.*

The error is distributed among the different courses in proportion to their length by the following

Rule.—*As the sum of all the courses is to any separate course, so is the whole difference in latitude to the correction for that course. A similar proportion corrects the departures.*

An example illustrating the process of balancing a survey is given below. In this example four separate columns are given for the corrected latitudes and departures. In practice, however, the corrected latitudes and departures are written in red ink directly above the original ones, which are crossed out with red ink. The distances given are in chains:

Stations.	Bearings.	Distances.	Latitudes.		Departures.		Corrected Latitudes.		Corrected Departures.	
			N +	S −	E +	W −	N +	S −	E +	W −
1	N 52° E	10.63	6.55		8.38		6.58		8.34	
2	S 29¾° E	4.10		3.56	2.03			3.55	2.01	
3	S 31¾° W	7.69		6.54		4.05		6.51		4.08
4	N 61° W	7.13	3.46			6.24	3.48			6.27
		29.55	10.01	10.10	10.41	10.29	10.06	10.06	10.35	10.35

The corrections are made by the following proportions:

For Latitudes.	For Departures.
29.55 : 10.63 :: 9 : 3 links.	29.55 : 10.63 :: 12 : 4 links.
29.55 : 4.10 :: 9 : 1 link.	29.55 : 4.10 :: 12 : 2 links.
29.55 : 7.69 :: 9 : 3 links.	29.55 : 7.69 :: 12 : 3 links.
29.55 : 7.13 :: 9 : 2 links.	29.55 : 7.13 :: 12 : 3 links.
9	12

This rule should not always be strictly followed, especially if one line has been measured over rough and broken country, while the others have been measured over smooth and open ground. In such a case the greater part of the error will probably lie in the rough line, and, consequently, it should receive the larger share of the correction. A slight alteration of a bearing will sometimes balance a survey. This may be done where an obstructed sight has probably caused an error in the bearing.

1332. Application of Latitudes and Departures to Platting.—Rule three columns, one for stations, the next for total latitudes, and the third for total departures, as shown in the following diagram.

To obtain the total latitudes, begin at any station, the extreme east or west one is preferable, and add up algebraically the latitudes of the following stations, observing that north latitudes are plus (+), and south latitudes minus (−). In the same manner find the algebraic sum of the departures for the different stations, placing each successive sum opposite its proper station.

In the example given in Art. **1330,** beginning at Station *1*, we obtain the following results.

Stations.	Total Latitudes from Station 1.	Total Departures from Station 1.
1	0.00	0.00
2	+ 2.26	+ 1.54
3	+ 2.36	+ 2.82
4	+ 1.15	+ 4.66
5	− 1.77	+ 2.67
1	0.00	0.00

The work is proved to be correct by the latitudes and departures for Station *1* coming out equal to 0. To apply these total latitudes and departures in platting, we draw a meridian through the point taken as Station *1*, Fig. 321. Scale off from Station *1* upwards on this meridian the latitude 2.26 chains to *A* and to the right from *A*, and perpendicularly lay off the departure 1.54 chains to Station *2*. Join *1-2*. From *1* again lay off the latitude 2.36 (= 2.26 + 10) chains to *B*, and to the right perpendicularly the departure 2.82 (= 1.54 + 1.28) chains to Station *3*. Join *2-3*, and proceed in like manner to locate Stations *4* and *5*, laying off + latitudes above Station *1* and + departures to the right of the meridian, and – latitudes below Station *1* and – departures to the left of the meridian. The principal advantages of this mode of platting are rapidity of work, the fact that each course is platted independently, and the certainty of the plats closing, provided the latitudes and departures have previously been balanced.

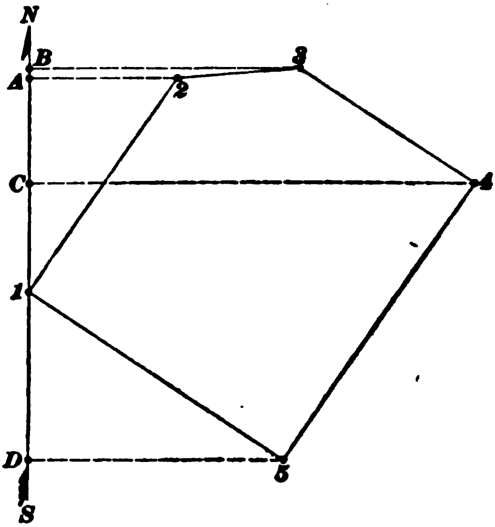


FIG. 321.

1333. Calculating the Content.—The survey of a field or farm having been made and platted, the content can always be found by dividing the plat into triangles, and scaling off their bases and perpendiculars from which the contents are calculated. This and other methods previously mentioned are only approximate, the degree of accuracy depending upon the largeness of the scale and the skill of the draftsman. The method of calculating content by latitudes and departures is perfectly accurate, and does not require the previous preparation of a plat.

1334. Definitions.—If a meridian be passed through the extreme east or west corner of a field, the perpendicular distance from any station to that meridian is the *longitude* of that station, additive or plus if east and subtractive or minus if west. The distance of the middle point of any line, such as the side of the field, from the meridian is called the **longitude** of that side. The difference of the longitudes of the two ends of a line is called the **departure** of the line; the difference of the latitudes of the two ends of a line is called the **latitude** of the line.

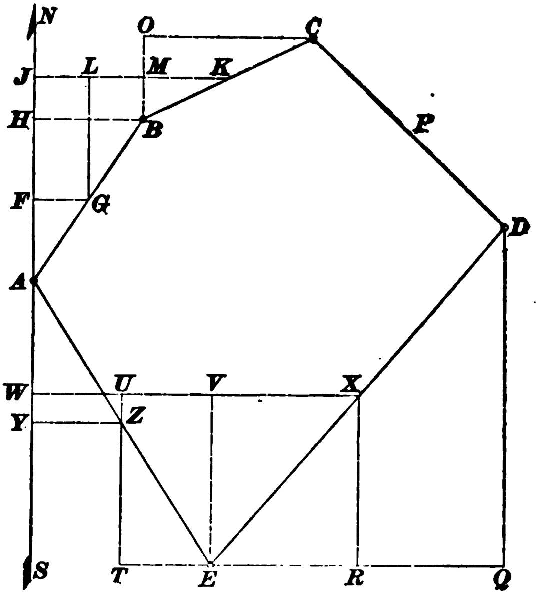


FIG. 322.

1335. Longitudes.—Let $N\ S$, Fig. 322, be the meridian passing through the extreme westerly station of the field $A\ B\ C\ D\ E$. From the middle and ends of each side draw perpendiculars to the meridian. These perpendiculars will be the longitudes and departures of the respective sides. The longitude $F\ G$ of the first course $A\ B$ is evidently equal to one-half its departure $H\ B$. The longitude $J\ K$ of the second course $B\ C$ is equal to $J\ L + L\ M + M\ K$ equal to the longitude of the first course plus half the departure of the first course plus half the departure of the course itself. The longitude $Y\ Z$ of some other course $E\ A$, taken anywhere, is equal to $W\ X - V\ X - U\ V$, or equal to the longitude of the preceding course minus half the departure of that course minus half the departure of the course itself, i. e., equal to the *algebraic* sum of these three parts, remembering that south latitudes and west longitudes are negative, and, therefore, to be subtracted when the instructions are to make an algebraic addition.

To avoid fractions, the preceding expressions are doubled, whence we deduce the following

Rule for Double Longitudes:

The double longitude of the first course is equal to its departure.

The double longitude of the second course is equal to the double longitude of the first course plus the departure of that course plus the departure of the second course.

The double longitude of the third course is equal to the double longitude of the second course plus the departure of that course plus the departure of the course itself.

The double longitude of any course is equal to the double longitude of the preceding course plus the departure of that course plus the departure of the course itself.

The double longitude of the last course (as well as of the first) is equal to its departure. This result, when obtained by the above rule, proves the accuracy of the calculation of the double longitudes of all the preceding courses.

1336. Areas.—The following is an application of the rule for finding areas by *double longitudes.* See Fig. 323. Let $A\ B\ C$ be a *three-sided field,* of which A is the most westerly station. Through A draw a meridian, and from the stations B and C and the middle points of the three sides of the field draw perpendiculars to the meridian. It is evident that the area of the field $A\ B\ C$ is equal to the area of the trapezoid $D\ B\ C\ E$ less the triangles $A\ D\ B$ and $A\ E\ C$. The area of the triangle $A\ D\ B$ is equal to the product of $A\ D$ by $F\ G$, i. e., it is equal to the product of the latitude of the first course by its longitude. The area of the trapezoid $D\ B\ C\ E$ is equal to the product of $D\ E$ by half the sum of $D\ B$ and $E\ C$, or $H\ K$, i. e., it is equal to the product of the latitude of the second course by its longitude. The area of the triangle $A\ E\ C$ is equal to the product of $A\ E$ by half $E\ C$, or $L\ M$, i. e., it is equal to the product of the latitude of the third course by its longitude. The bearing of the course $A\ B$ is N E, and that of $C\ A$ is N W. Their latitudes are, therefore, *north.* The bearing of the course $B\ C$ is S E, and its latitude is *south.* Calling the products in which the latitude is north, *north products,* and the products in which the latitude is south, *south products,* we find the area of the trapezoid to be a south product and the areas of the triangles to be north products. The difference of the north products and the south products is, therefore, the area of the three-sided field $A\ B\ C$.

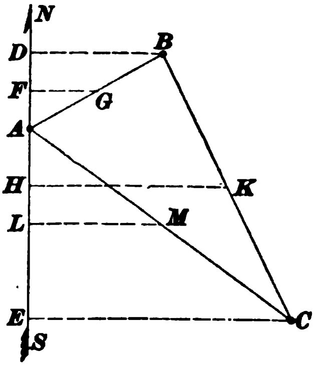


FIG. 323.

Using *double* longitudes, to avoid fractions, in each of the preceding products, their difference will be *double* the area of the field $A\ B\ C$.

Take, now, a four-sided field, $A\ B\ C\ D$, Fig. 324, and drawing a meridian through its most westerly station A, and longitudes as in the preceding case, it will be evident from inspection that the area of the field $A\ B\ C\ D$ is equal to the trapezoid $F\ C\ D\ G$, diminished by the area of triangles $A\ G\ D$, $A\ E\ B$, and the trapezoid $E\ B\ C\ F$. The area of the triangle $A\ E\ B$ is equal to the product of the latitude $A\ E$ of the first course by its longitude $H\ K$. Its product is *north*. The area of the trapezoid $E\ B\ C\ F$ is equal to the product of the latitude $E\ F$ of the second course by its longitude $L\ M$, and is also a *north* product. The area of the trapezoid $F\ C\ D\ G$ is equal to the product of the latitude $F\ G$ of the third course by its longitude $N\ O$, a *south* product. The area of the triangle $A\ G\ D$ is equal to the product of the latitude $A\ G$ of the fourth course by its longitude $P\ Q$, a north product. Subtracting the sum of the north products from the sum of the south products the difference is the area of the field $A\ B\ C\ D$. If *double* longitudes had been used, as in the previous case, the difference would have been *double* the area of the field.

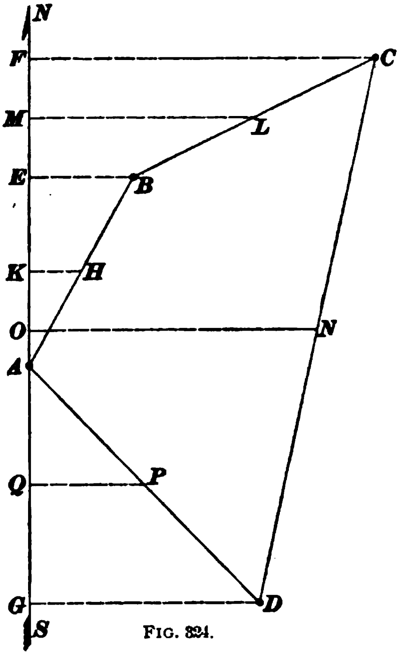


FIG. 824.

1337. The Application of Double Longitudes to the Finding of Areas.—Whatever the number or directions of the sides of a field or any surface enclosed by straight lines, its area will always be equal to *half* the difference of the north and south products arising from multiplying together the latitude and *double* longitude of each course or side, whence the following

General Rule for Finding Areas:

1. Prepare ten columns, headed as in the following examples, and in the first three write the stations, bearings, and distances.

2. Find the latitudes and departures of each course by the traverse table, as directed in Art. ***1329,*** *placing them in the four following columns.*

3. Balance them as in Art. ***1331,*** *correcting them in red ink.*

4. Find the double longitudes as in Art. ***1335,*** *with reference to a meridian passing through the extreme east or west station, and place them in the eighth column.*

5. Multiply the double longitude of each course by the corrected latitude of that course, placing the north products in the ninth column and the south products in the tenth column.

6. Add the last two columns; subtract the smaller sum from the larger, and divide the difference by two. The quotient will be the content required.

1338. To Find the Most Easterly or Westerly Station of a Survey.—Make a rough hand sketch of the tract, giving the sides, their approximately true direction, and length. The most easterly or westerly station may then be determined from an inspection of the sketch.

Example 1 of this article refers to the five-sided field, a plat of which is given in Fig. 321, and the latitudes and departures of which were calculated in Art. **1330.** Station *1* is the most westerly in the plat, and the meridian will be passed through it.

The double longitudes are found by applying the rule for double longitudes, given in Art. **1335.** As the additions are made algebraically, due attention must be paid to the signs. The double longitudes are marked D. L., as shown in the marginal diagram. These double longitudes are obtained by the following operation. As stated in the rule, the double longitude of the first course is equal to the departure. By reference to the given example, we find that the departure of the first course is 1.54 chains, an *east* departure, and, therefore, positive. We record this in the column headed D. L., opposite Station *1*. The D. L. of the second course is equal to the D. L. of the first course plus the departure of that course plus the departure of the second course. Accordingly, we place under the D. L. of the first course, the departure of that course, viz., + 1.54, and the departure of the second course, viz., + 1.28, as given in the east departure column of the example. This sum, viz., + 4.36 is the D. L. of the second course and placed opposite Station *2*. The D. L. of the third course is equal to the D. L. of the second course plus the departure of that course plus the departure of the third course. Accordingly, we place under the D. L. of the second course the departure of that course, viz., + 1.28, and the departure of the third course, viz., + 1.84. This sum, + 7.48, is the D. L. of the third course, which we place opposite Station *3*. In a similar manner we find the D. L. for the fourth and fifth courses. The double longitude of the last course is equal to its departure, which proves the work. The double longitudes of the courses are then multiplied by their corresponding latitudes, and the content of the field obtained as directed in the given rule.

Stations.	D. L.
1	+ 1.54 D.L.
	+ 1.54
	+ 1.28
2	+ 4.36 D.L.
	+ 1.28
	+ 1.84
3	+ 7.48 D.L.
	+ 1.84
	− 1.99
4	+ 7.33 D.L.
	− 1.99
	− 2.67
5	+ 2.67 D.L.

Had the meridian been supposed to pass through Station *4*, the most easterly station, all the longitudes would have been west or minus, but the difference in the double areas would have been the same, giving the same content as before.

The following examples will give the student some practice in the use of traverse tables, and in applying latitudes and departures in the calculation of areas:

EXAMPLE 1.

Stations.	Bearings.	Distances. Chains	Latitudes.		Departures		Double Longitudes.	Double Areas.	
			N +	S −	E +	W −		N +	S −
1	N 34¼° E	2.73	2.26		1.54		+ 1.54	3.4804	
2	N 85½° E	1.28	.10		1.28		+ 4.36	.4360	
3	S 56¾° E	2.20		1.21	1.84		+ 7.48		9.0508
4	S 34¼° W	3.53		2.92		1.99	+ 7.33		21.4036
5	N 56½° W	3.20	1.77			2.67	+ 2.67	4.7259	
		12.94	4.13	4.13	4.66	4.66		8.6423	30.4544
									8.6423
									2)21.8121

Content = 1 A. 0 R. 14.50 P.

Square chains, 10.9061

EXAMPLE 2.

Stations.	Bearings.	Distances. Chains.	Latitudes.		Departures.		Double Longitudes.	Double Areas.		Stations.	D. L.
			N +	S −	E +	W −		N +	S −		
			6.56		8.37						
1	N 52° E	10.64	~~6.55~~		~~8.38~~		+ 8.37	54.9072		1	+ 8.37 D. L.
2	S 29¾° E	4.09		3.55	2.02		+ 18.76		66.5980		+ 8.37
				6.52							+ 2.02
						4.05					
3	S 31¾° W	7.68		~~6.53~~		~~4.04~~	+ 16.73		109.0796	2	+ 18.76 D. L.
						6.34					+ 2.02
4	N 61° W	7.24	3.51			~~6.33~~	+ 6.34	22.2534			− 4.05
		29.65	10.06	10.08	10.41	10.37		77.1606	175.6776	3	+ 16.73 D. L.
			10.07	10.07	10.39	10.39			77.1606		− 4.05
									2) 98.5170		− 6.34
										4	+ 6.34 D. L.

Content = 4 A. 3 R. 28.14 P.

Square chains, 49.2585

EXAMPLE 3.

Stations.	Bearings.	Distances. Chains.	Latitudes.		Departures.		Double Longitudes.	Double Areas.	
			N +	S −	E +	W −		N +	S −
1	S 57° E	5.77		3.13 ~~3.14~~	4.85 ~~4.84~~		− 4.85		15.1805
2	S 36¼° W	2.25		1.81		1.33	− 1.33		2.4073
3	S 39¼° W	1.00		.77		.63	− 3.29		2.5333
4	S 70¼° W	1.04		.35		.98	− 4.90		1.7150
5	N 68¾° W	1.23	.45			1.15	− 7.03	3.1635	
6	N 56° W	2.19	1.22			1.82	− 10.00	12.2000	
7	N 33¼° E	1.05	.88		.58		− 11.24	9.8912	
8	N 56½° W	1.54	.85			1.28	− 11.94	10.1490	
9	N 33½° E.	3.18	2.66 ~~2.65~~		1.76		− 11.46	30.4836	
		19.25	6.05	6.07	7.18	7.19		65.8873	21.8361
			6.06	6.06	7.19	7.19		21.8361	
								2) 44.0512	
							Square chains,	22.0256	

Content = 2 A. 0 R. 32.4 P.

Stations.	D. L.
2	− 1.33 D. L. − 1.33 − .63
3	− 3.29 D. L. − .63 − .98
4	− 4.90 D. L. − .98 − 1.15
5	− 7.03 D. L. − 1.15 − 1.82
6	− 10.00 D. L. − 1.82 + .58
7	− 11.24 D. L. + .58 − 1.28
8	− 11.94 D. L. − 1.28 + 1.76
9	− 11.46 D. L. + 1.76 + 4.85
1	− 4.85 D. L.

The notes of the survey given in Example 3 are platted by total latitudes and total departures from Station *1*. A plat of the survey is given in Fig. 325 and the total latitudes and departures in the accompanying table. From an inspection of the plat it will be seen that Station *2* is the most easterly, and the double longitudes given in Example 3 are reckoned from a meridian passing through that station.

Stations.	Total Latitudes from Station 1.	Total Departures from Station 1.
1	0.00	0.00
2	− 3.13	+ 4.85
3	− 4.94	+ 3.52
4	− 5.71	+ 2.89
5	− 6.06	+ 1.91
6	− 5.61	+ .76
7	− 4.39	− 1.06
8	− 3.51	− .48
9	− 2.66	− 1.76
1	− 0.00	0.00

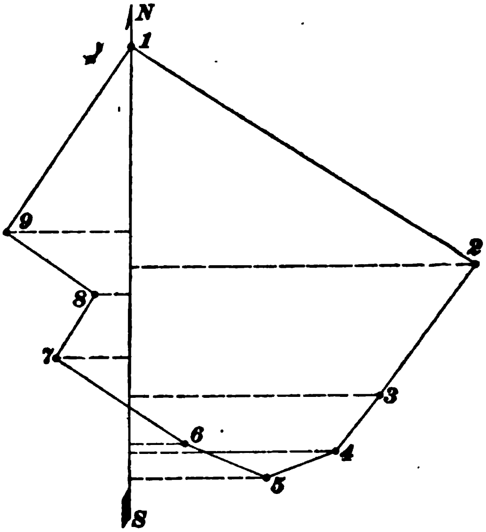


FIG. 325.

TOWN SITES AND SUBDIVISIONS.

1339. First Considerations.—In laying out town sites the consideration of first importance is the location of the streets rather than the greatest number of lots obtainable. The custom of laying out town sites in rectangular lines, without reference to topographical conditions, prevails almost universally throughout the United States. This is largely owing to two principal causes, viz., first, the supposition that the rectangular method or plan will yield the greater number of lots, and, hence, the greater profit, and, second, the haste in surveying, platting, and placing the property on the market does not admit of a thorough study of the ground.

The town site should be considered as a whole, the location of its main streets and thoroughfares being determined by traffic considerations chiefly. These considerations will necessarily involve the questions of grades, drainage, and railway communications. Without the latter there is small excuse for a town. Where possible a main avenue should be laid out, parallel with the railroad, leaving one tier of lots between the street and track. This avenue may then be used as a base from which to lay out the adjoining streets and avenues, which may be parallel with and at right angles to it if the surface be generally level, or at oblique angles if the surface be rolling or hilly.

1340. Grades, Drainage, and Topography.—Grades and drainage should be so arranged that surface water will tend to form main channels, i. e., the surface water of several streets will find its way into some particular street where special provision can be made for its control and discharge.

The streets in the residential portion of the town should, so far as possible, conform to the existing topographical conditions. This will greatly reduce the cost of grading the streets, give easy grades, and so promote comfort of pedestrians and the efficiency of teams. There is no loss in frontage from the employment of curves instead of straight lines, and there is no question of the advantage of the former from an artistic standpoint.

The accompanying plan, Fig. 326, is made to meet the following conditions, viz., two lines of railroad, a main line and branch, meet at the junction of two streams. The land bordering on the smaller stream, which is followed by the branch road, rises rapidly from the stream, reaching a height of 200 feet, and then falls gradually until an elevation of 50 feet above the stream is reached, when the surface remains generally level. The land bordering the larger stream, which is followed by the main line of the railroad, rises gradually until a height of 50 feet above the stream is reached, beyond which the surface is generally level.

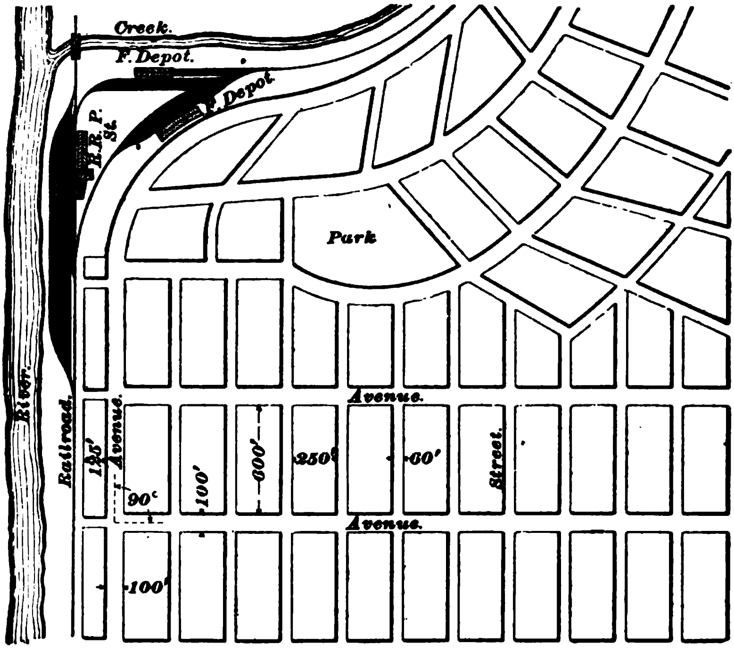


Fig. 326.

That part of the given surface which is generally level will be laid out in rectangular blocks. The unaided eye will readily determine whether a town site is well adapted to rectangular divisions. If it is so adapted the order of survey will be as follows:

1341. General Directions for Preliminary Survey.—Run a line enclosing the entire area, giving location of prominent features, such as railroads, highways, streams, houses, etc., and accurately plat to a scale of 200 feet to the inch. For cities, avenues are made 100 feet in width between building lines, and streets 60 feet, the avenues being parallel to each other and the streets at right angles to the avenues. City lots usually have fronts of 25 feet and depths of 125 feet. Part of New York is laid out in blocks of 200 feet by 800 feet, the 200 feet facing the avenues. Lots are 100 feet in depth, each block containing 64 lots. Having determined the dimensions of the streets and blocks, lay out the principal base line so that it will form the center line of a street or avenue running parallel with the general direction of the railroad, providing for overhead or sub-crossings where practicable. If crossings must be at grade, the fewer of them the better. Provide easy and safe access to railroad stations and freight depots.

Lay out the plat in rectangular blocks, accurately scaling all dimensions. Arrange the plan so as to interfere as little as possible with existing lines of travel, at the same time giving due regard to the future needs of an increasing population. If the ground is wooded or sight obstructed by underbrush, but one additional base line can be used. It should be about midway between the extremities of the principal base line and at right angles to it. If, however, the ground is open with nothing to interfere with long sights, two base lines should be laid out, one at either extremity of the main base and at right angles to it.

1342. Measurements.—All measurements should be made with a standard steel tape and plumb-bob and carefully checked. The base lines especially should be measured with great care, as the correctness of all the subsequent measurements depends upon the degree of accuracy with which these primary lines are measured.

1343. Base Lines and Subdivisions.—The rectangular method of surveying town sites is illustrated in Fig. 327, in which *A B* is the principal base, and the auxiliary bases *A D* and *B C* are laid off from the extremities of *A B*. The avenues are at right angles to *A B*, and the streets parallel to *A B*. Avenue *A* is parallel to the railroad,

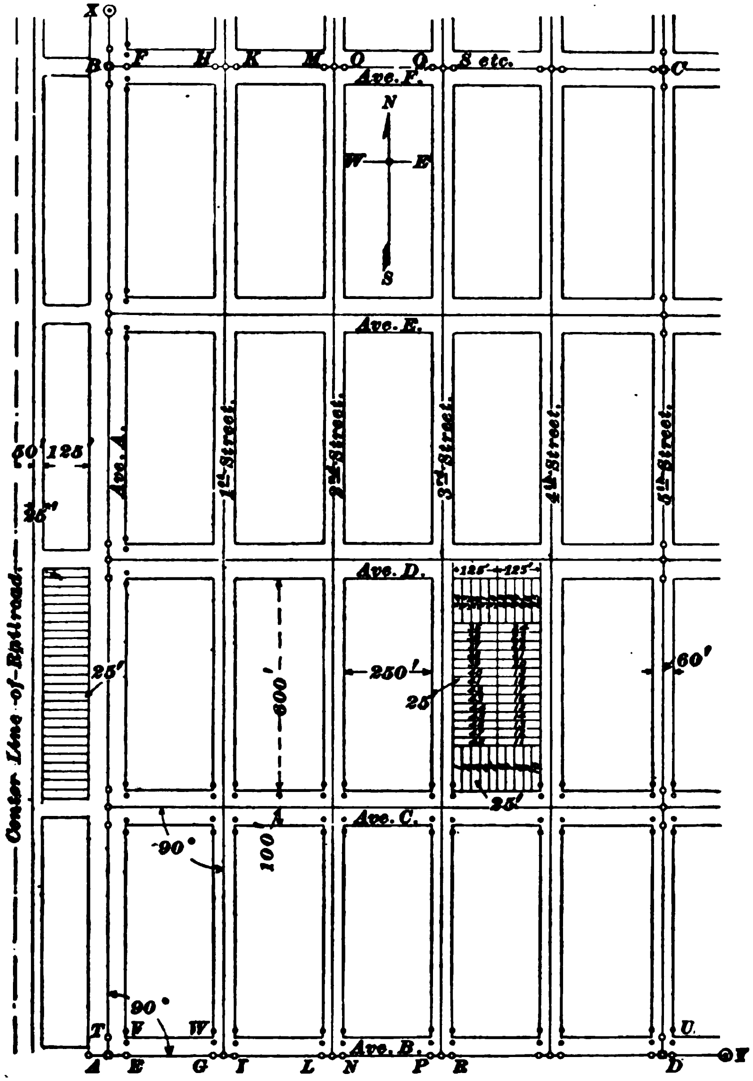


FIG. 327.

from which it is separated by a 25-foot alleyway and one tier of lots 125 feet deep. Avenues are 100 feet in width, streets 60; blocks 250 feet by 600 feet, fronting 250 feet on the avenues, and the lots are 25 feet by 125 feet.

The initial point *A* of the principal base *A B* is the center of an avenue, and should be fixed by a plug 2′ × 2′ × 18′ driven flush with the surface of the ground and the center marked by a tack, with a guard stake beside it, and numbered 0. Drive a temporary plug at *X* to be used as a foresight in giving the direction of *A B*. Set up the instrument at *A* and sight to *X*, frequently checking the foresight. Measure from *A* on *A B* 50 feet, and drive a 12′ plug, carefully centering the same. This point marks the north side of Avenue *B*. Continue measuring on the line *A B*, driving a stake at each hundred feet, marking the exact measurement by a tack, and number in regular succession from *A*. At 650 feet from *A* set a 12′ plug with tack center, marking the south side of Avenue *C*; 50 feet further, at Station 7, set a 12′ plug with center. This point will be in the center of Avenue *C*; at 7 + 50 a plug with center is set, marking the north side of Avenue *C*. In like manner locate the center and sides of all avenues lying between *A* and *X*, always checking the foresight before setting tack centers. At *B* set a plug 2′ × 2′ × 18′. At each station remeasure the last 100 feet, so as to secure accurate results. The measurement of the main base line *A B* being completed, take a foresight on *X* and turn off an angle of 90°, setting a temporary plug at *Y*. Mark the center at *Y* with a pencil point and repeat the angle five times, marking a center at *Y* for each angular measurement. These points will vary slightly in position, though two of them may fall at the same place. Take the mean or average of these points, and mark the point with a tack. Then, commencing at *A* measure the line *A Y*, setting stakes at each 100 feet, as in *A B*, and set hubs on the side lines of Avenue *A* and at centers and side lines of the streets parallel to it, checking the foresight at *Y* and the measurement of each station before setting plug centers. On this line the measurements will be, first 50 feet, next 250, then 60, 250, 60, etc., the streets being 60 feet and the blocks 250 feet in width. At *D* set a plug

2′ × 2′ × 18′. In a similar manner locate the base line *B C* and the street centers and side lines on *B C*. Then set up at *D*, which is a permanent point, and foresighting to *C* set points on centers and sides of avenues on *D C*. Next set up at *E*, foresighting to *F*. Provide a supply of plugs 1′ × 1′ × 8′ and measure from *E*, 50 feet in both directions on the line *E F*, and mark the points with pins. These points will be on the north and south lines of Avenue *B*. On either side of these pins in the line *E F*, and about two feet from them, set plugs, leaving two inches of their length above the surface of the ground. Center these plugs, driving tacks half their length into each one. In the same manner set plugs on both sides of each street line as indicated by the small circles in the figure. In this example *B C* is 2,800 feet distant from *A D*, too great a distance for the accurate setting of plug centers; therefore, the measurements from *A D* to *B C* should terminate on the south line of Avenue *D*. In the same manner locate plugs on the lines *G H*, *I K*, *L M*, etc. Having set all plugs between the base *A D* and the south line of Avenue *D*, move the instrument to *F* and, foresighting to *E*, set plugs on both sides of all avenues between Avenue *F* and Avenue *D*, including the north side of *D*. In like manner locate plugs on *H G*, *K I*, *M L*, etc. Move the instrument to the point *U*; stretch pieces of cord between the plugs on both sides of the line *X T*; foresight to *T*, and at each intersection with the cord, as at *V*, *W*, etc., drive a 12′ plug, and center with a tack. This method of locating street corners by intersections has the advantage of bringing all corners on the same street in perfect line, a result which it would be practically impossible to obtain by direct measurement. The measurement of all angles is referred to the base lines where special care is taken in checking them.

1344. Permanent Monuments.—The street and avenue centers located on the base lines should always be rendered permanent by setting stone monuments at those points.

MAPPING.

INTRODUCTION.

1345. The object of this section is to furnish the student thorough, practical instruction in mapping. Having previously mastered the section on Geometrical Drawing, he should by this time be familiar with the various instruments employed in the drafting room, and be accustomed to their use.

All the principles and methods here described are fully illustrated by drawings, which comprise six plates, found at the end of the volume on Geometrical Drawing. These plates the student will be required to draw, and the degree of proficiency, as shown by his work, will determine his standing. The examples given in the plates are similar to those met with in practical field and office work.

1346. A **map** is a series of lines and angles so combined as to represent the true outlines, proportions, and character of any required surface.

1347. **Lines** are either boundaries or divisions of the required surface. They have only the properties of direction and length.

DRAWING THE PLATES.

PLATE, TITLE: PLATTING ANGLES I.

1348. This plate contains six angle lines, three of which are comprised by Fig. 1 and three by Fig. 2. The three lines *a*, *b*, and *c*, under Fig. 1, will be drawn to a scale of 200 feet to the inch, platting the angles with a protractor, the

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PLATTING AN

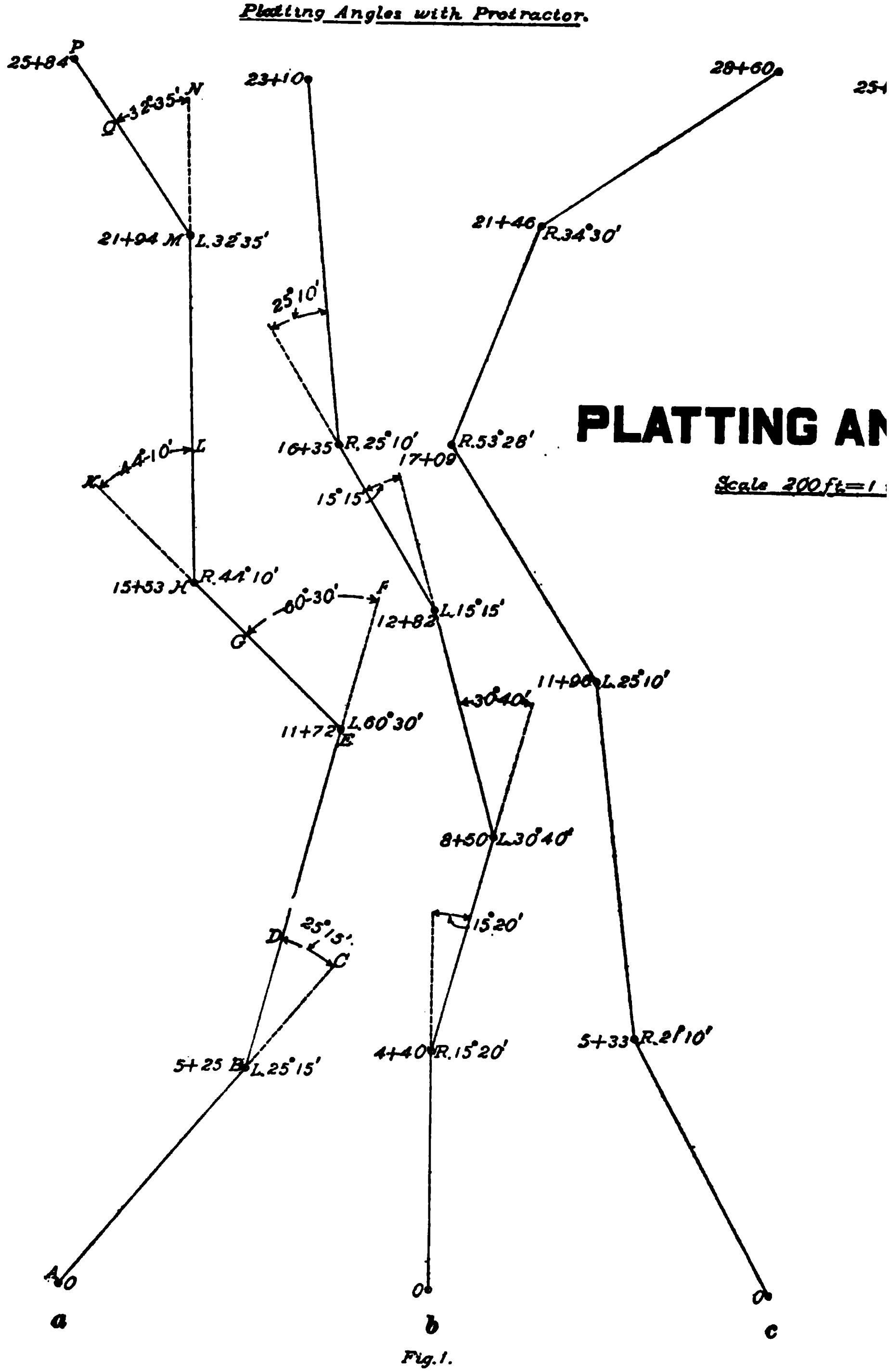


Fig. 1.

ANGLES.-I

=1 in.

Platting Angles by Chords.

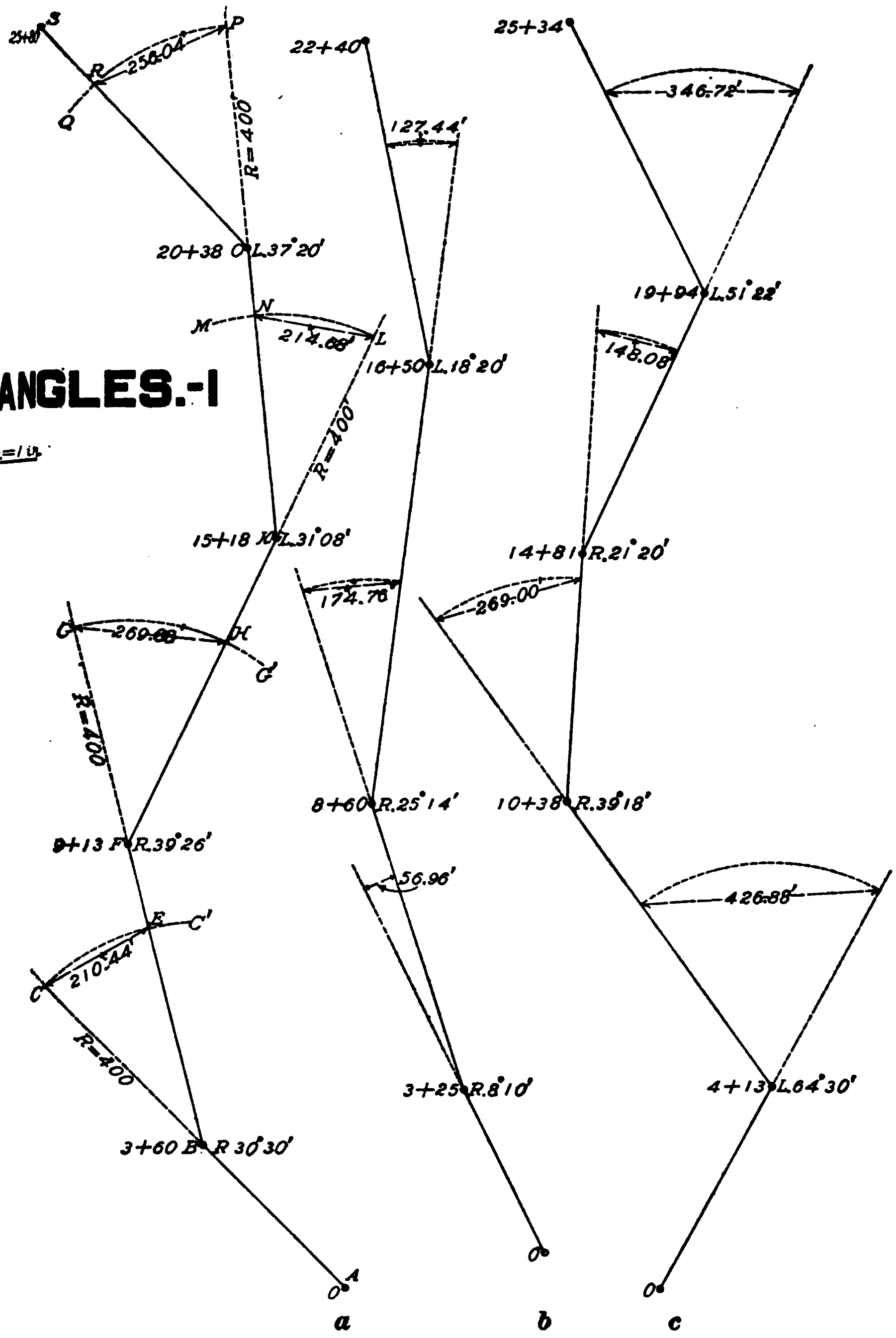


Fig. 2.

age immediately following the title page. JOHN SMITH, CLASS Nº 4529.

placing the center of the protractor on the point *B*, with the zero point on the line *B C*, lay off the angle 25° 15′ to the left of *B C*, marking the point of angle measurement *D* with a needle point. Through the points *B* and *D* draw a straight line. The angle *C B D* is 25° 15′, and the line *B D* is the direction of the next course. The second angle, 60° 30′, is turned to the left at Sta. 11 + 72. The length of the second course is found by subtracting 525 from 1,172, giving a difference of 647 ft. Produce *B D* and scale off the second course 647 ft., locating the point *E* at Sta. 11 + 72. Produce *B E* to *F*, and lay off to the left of *E F* the angle 60° 30′, locating the point *G*. Join *E* and *G*. The angle *F E G* is 60° 30′, and the line *E G* is the direction of the next course.

The third angle is R. 44° 10′, and is turned at Sta. 15 + 53. The length of the third course is found by subtracting 1,172 from 1,553, giving a difference of 381 feet. Produce *E G* and scale off from *E* the distance 381 ft., locating the point *H* at Sta. 15 + 53. Produce *E H* to *K*, and to the right of *H K* lay off the given angle 44° 10′, locating the point *L*. The line joining the points *H* and *L* forms with *H K* an angle of 44° 10′, and gives the direction of the next course. The next angle is L. 32° 35′, and is turned at Sta. 21 + 94. The length of the course is found by subtracting 1,553 from 2,194, giving a difference of 641 ft. Produce *H L* and scale off from *H* the distance 641 ft., locating the point *M* at Sta. 21 + 94. Produce *H M* to *N*, and to the left of *M N* lay off the given angle 32° 35′, locating the point *O*. Draw *M O*. The angle *N M O* is 32° 35′, and *M O* is in the direction of the next and last course of line *a*, whose length is found by subtracting 2,194 from 2,584. The difference is 390 ft. We produce the line *M O*, and from *M* scale off the last course of 390 ft., locating the point *P* at Sta. 25 + 84. At each angular point in the line an arc is described, giving the measurement of the angle.

The student will in a similar manner plat the following notes for the lines *b* and *c*, Fig. 1, of the same plate:

NOTES FOR LINE *b*.

Stations.	Angles.
23 + 10	End of Line
16 + 35	R. 25° 10′
12 + 82	L. 15° 15′
8 + 50	L. 30° 40′
4 + 40	R. 15° 20′
0	

NOTES FOR LINE *c*.

Stations.	Angles.
28 + 60	End of Line
21 + 46	R. 34° 30′
17 + 09	R. 53° 28′
11 + 96	L._25° 10′
5 + 33	R. 21° 10′
0	

1350. To Lay Off an Angle by Chords.—This is done by means of a table of chords in which the lengths of chords for all angles from 0 to 90° are given in terms of a radius 1. A radius of any convenient length may be assumed, and the corresponding chord length obtained by multiplying the length of the chord given in terms of radius 1 by the length of the assumed radius. Thus, let it be required to lay off from a given line an angle of 40° 10′ to the left. Let *A B*, Fig. 328, be the given line. Produce *A B* to *C*, making *B C* = 400 ft., the length of the assumed radius.

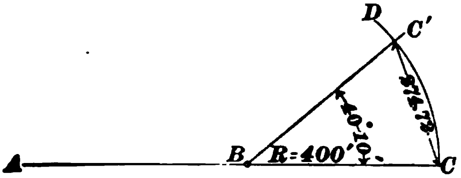


FIG. 328.

From a table of chords, we find that the chord of an angle of 40° 10′ in terms of a radius 1 is .6868. Multiplying this chord by 400 ft., the length of the assumed radius, we have 274.72 ft., the length of the required chord. From *B* as a center, with a radius *B C* = 400 ft., describe to the left of *B C* the indefinite arc *C D*, being sure that the length of *C D* is slightly greater than the length of the required chord, and from *C* as a center, with a radius of 274.72 ft., describe an arc intersecting the arc *C D* in *C′*. Through

B and *C'* draw a straight line. The angle *C B C'* is 40° 10', the required angle. This method of platting angles is more accurate, though less rapid, than platting with a protractor.

NOTE.—The table of chords used for the calculations given in this Course may be found in Trautwine's Pocket Book, a very useful book to all surveyors. If the student does not possess a copy, he may easily find the required chord from his table of sines by multiplying the sine of half the given angle by 2. Thus, the chord of 40° 10' $= 2 \sin \frac{40^\circ\ 10'}{2} =$ $2 \sin 20^\circ\ 05' = 2 \times .34339 = .68678 = .6868$, using but four places, the same as given in the table.

1351. Fig. 2, same plate as above, contains three examples in the lines *a*, *b*, and *c*, in which the angles are laid off by chords. The notes for example *a* are given in the accompanying table.

The first course *A B* is 360 feet in length, which the student will draw to a scale of 200 feet to the inch. The starting point *A* is numbered 0, and *B*, the end of the first course, 3+60. At *B* an angle of 30° 30' is laid off to the right. Produce *A B* 400 feet, which we assume to be the length of the radius in calculating chord lengths for laying off angles, and locate the point *C*. Then, from *B* as a center, with a radius of 400 feet, describe the indefinite arc *C C'* on the right side of the radius *B C*, being sure that the arc shall contain at least 30° 30'. We find in a *table of chords* that the chord of 30° 30' = .5261, which, multiplied by 400 ft., the length of the assumed radius, gives 210.44 ft., the length of the required chord. From *C* as a center, with a radius of 210.44 ft., describe an arc intersecting the arc *C C'* in the point *E*. A line joining *B* and *E* will form with the radius *B C* an angle *C B E* = 30° 30', the required

NOTES FOR LINE *a*.

Stations.	Angles.
25 + 80	End of Line
20 + 38	L. 37° 20'
15 + 18	L. 31° 08'
9 + 13	R. 39° 26'
3 + 60	R. 30° 30'
0	

angle. The next angle R. 39° 26′ is turned at Sta. 9 + 13, making the length of the second course 553 ft. Denote Sta. 9 + 13 by F. Produce $B\ F$ 400 ft. to G. From F as a center, with a radius $F\ G$ of 400 ft., describe to the right of $F\ G$ the indefinite arc $G\ G'$, being sure that the arc shall contain at least 39° 26′. The chord of 39° 26′ to a radius 1 is .6747, which, multiplied by 400 ft., gives 269.88 ft., the length of the required chord. From F as a center, with a radius of 269.88 ft., describe an arc intersecting the arc $G\ G'$ in H. A line joining F and H will form with the radius $F\ G$ the angle $G\ F\ H = 39^\circ\ 26'$, the required angle. The next angle, viz., L. 31° 08′, is turned at Sta. 15 + 18, making the length of the third course 605 ft. Call Sta. 15 + 18, K. Produce $F\ K$ 400 ft. to L. From K as a center, with a radius $K\ L$ of 400 ft., describe to the left of $K\ L$ the indefinite arc $L\ M$. The chord of 31° 08′ is .5367, which, multiplied by 400 ft., gives 214.68 ft., the length of the required chord. From L as a center, with a radius of 214.68 ft., describe an arc intersecting the arc $L\ M$ in the point N. Join K and N, forming with $K\ L$ the angle $L\ K\ N = 31^\circ\ 08'$. The next angle, viz., L. 37° 20′, is turned at Sta. 20 + 38, making the length of the fourth course 520 ft. Call Sta. 20 + 38, O. Produce $K\ O$ 400 ft. to P. From O as a center, with a radius $O\ P$, describe the indefi-

NOTES FOR LINE *b*.

Stations.	Angles.
22 + 40	End of Line.
16 + 50	L. 18° 20′
8 + 60	R. 25° 14′
3 + 25	R. 8° 10′
0	

NOTES FOR LINE *c*.

Stations.	Angles.
25 + 34	End of Line.
19 + 94	L. 51° 22′
14 + 81	R. 21° 20′
10 + 38	R. 39° 18′
4 + 13	L. 64° 30′
0	

nite arc $P\,Q$. The chord of 37° 20′ is .6401, which, multiplied by 400 ft., gives 256.04 ft., the length of the required chord. From P as a center, with a radius of 256.04 ft., describe an arc intersecting the arc $P\,Q$ in R. Join O and R, forming with $O\,P$ the angle $P\,O\,R = 37° 20'$. The end of the line S is at Sta. 25 + 80, making the length of the last course 542 ft. In a similar manner, plat the notes for lines *b* and *c*, which are given in Art. **1353.**

1352. To Lay Off an Angle by its Bearing.—By this method of laying off angles, the direction of each line is referred to the magnetic meridian, which maintains a constant direction, being a north and south line. The bearing of a line is the angle which the line makes with the magnetic meridian. In platting a land or railroad survey, a pencil

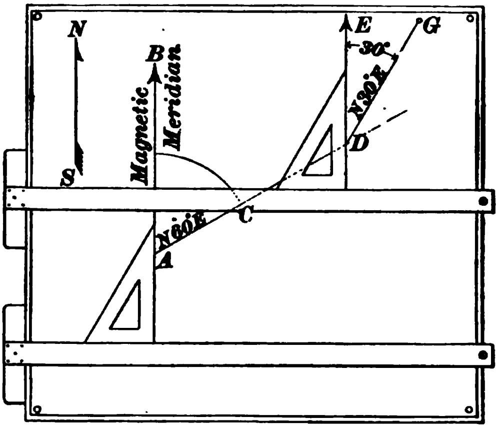


FIG. 329.

line giving the direction of the magnetic meridian is drawn through each station at which a bearing is taken.

The direction of the meridian may be given by means of the ordinary **T** square and triangles, and the angles laid off either by a protractor or by tangents. The use of **T** square and triangles in laying off angles by bearings is illustrated in Fig. 329. A sheet of paper is fastened to a drawing board. It is well known that if the head of the **T** square be kept firmly pressed against the side of the drawing board, as shown in the figure, the lines drawn along the straight edge will be parallel; hence, the lines drawn perpendicular to this straight edge by means of the triangles, as shown in the figure, will be parallel.

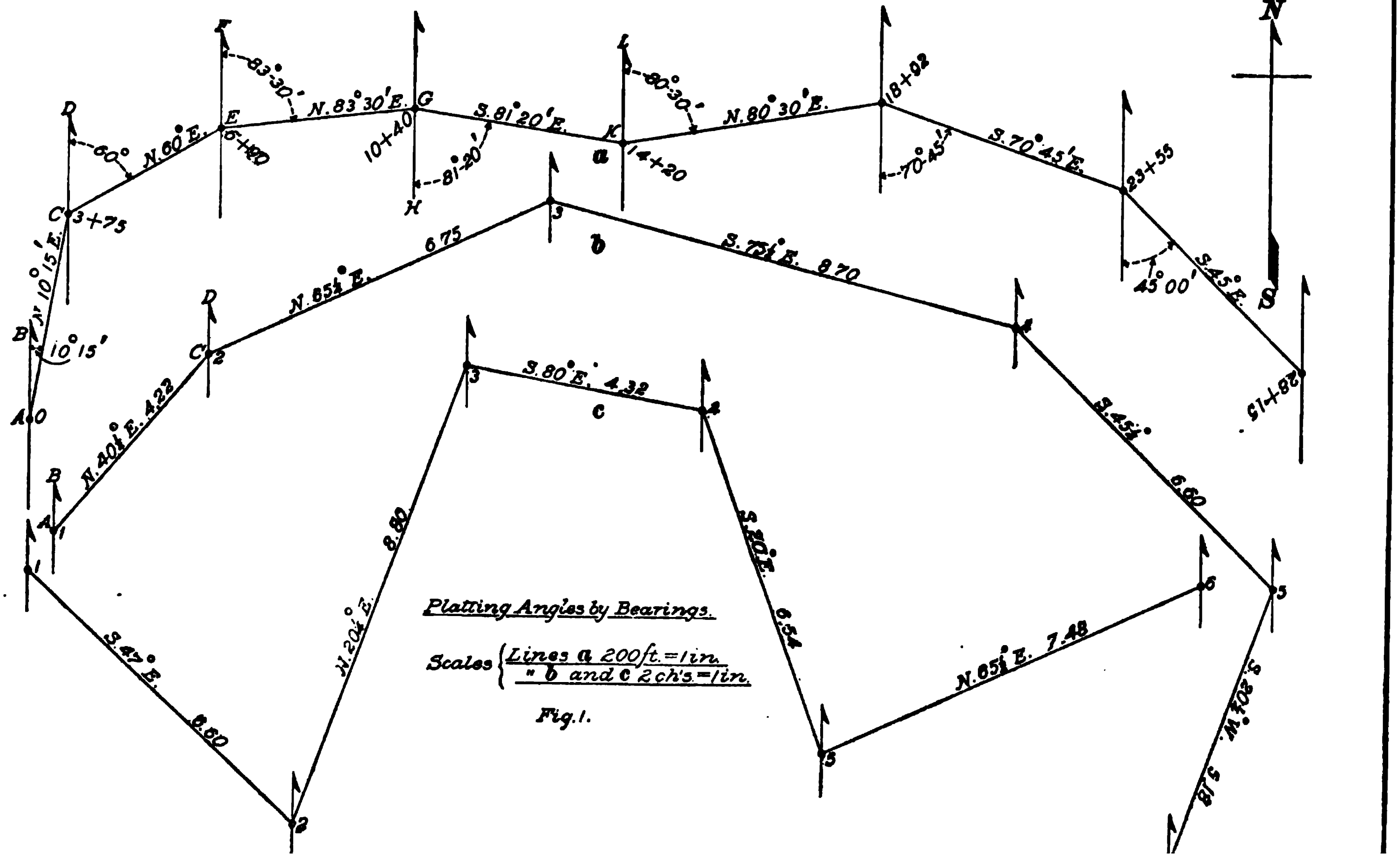
Either the parallels drawn along the straight edge of the **T** square or of the triangle may be used as the magnetic meridian, though the latter is preferable, as it brings the north end of the meridian at the top of the map, which is its proper position.

Let it be required to plat a line having a bearing of N 60° E. As in Fig. 329, a point *A* is assumed as the station at which the bearing is taken. Through the point *A*, a line *A B* is drawn perpendicular to the straight edge of the **T** square. This line will represent the direction of the magnetic meridian. As the bearing is east, the angle of 60° will be to the right of *A B*. Place a protractor with its center at *A* and its zero point in the line *A B*. Lay off the angle 60°, and mark the point of measurement *C* with a needle point. Draw a line joining the starting point *A* with the point of angle measurement *C*. The line *A C* will then form an angle of 60° with the meridian, and its course will be N 60° E. From the notes find the length of the first course, and measure on the line *A C* to some convenient scale the length of that course, locating the point *D*, where the next bearing N 30° E is taken. Slide the **T** square upwards, and with the triangle draw through *D* another meridian *D E*. From *D* as a center lay off from the right of the meridian *D E* the bearing N 30° E. Let *F* mark the measurement of this angle. The line joining *D* and *F* will have a bearing of N 30° E.

PLATE, TITLE: PLATTING ANGLES II.

1353. This plate contains five angle lines, the angles of the three lines given in Fig. 1 being platted by *magnetic bearings*, and those in Fig. 2 by *tangents*. In Fig. 1, line *a*, the distances are given in stations of 100 feet each; in the lines *b* and *c*, the distances are given in chains. The student

PLATTING ANGLES.-II



Platting Angles by Bearings.

Scales { *Lines* **a** *200 ft. = 1 in.*
*" **b** and **c** 2 ch's. = 1 in.*

Fig. 1.

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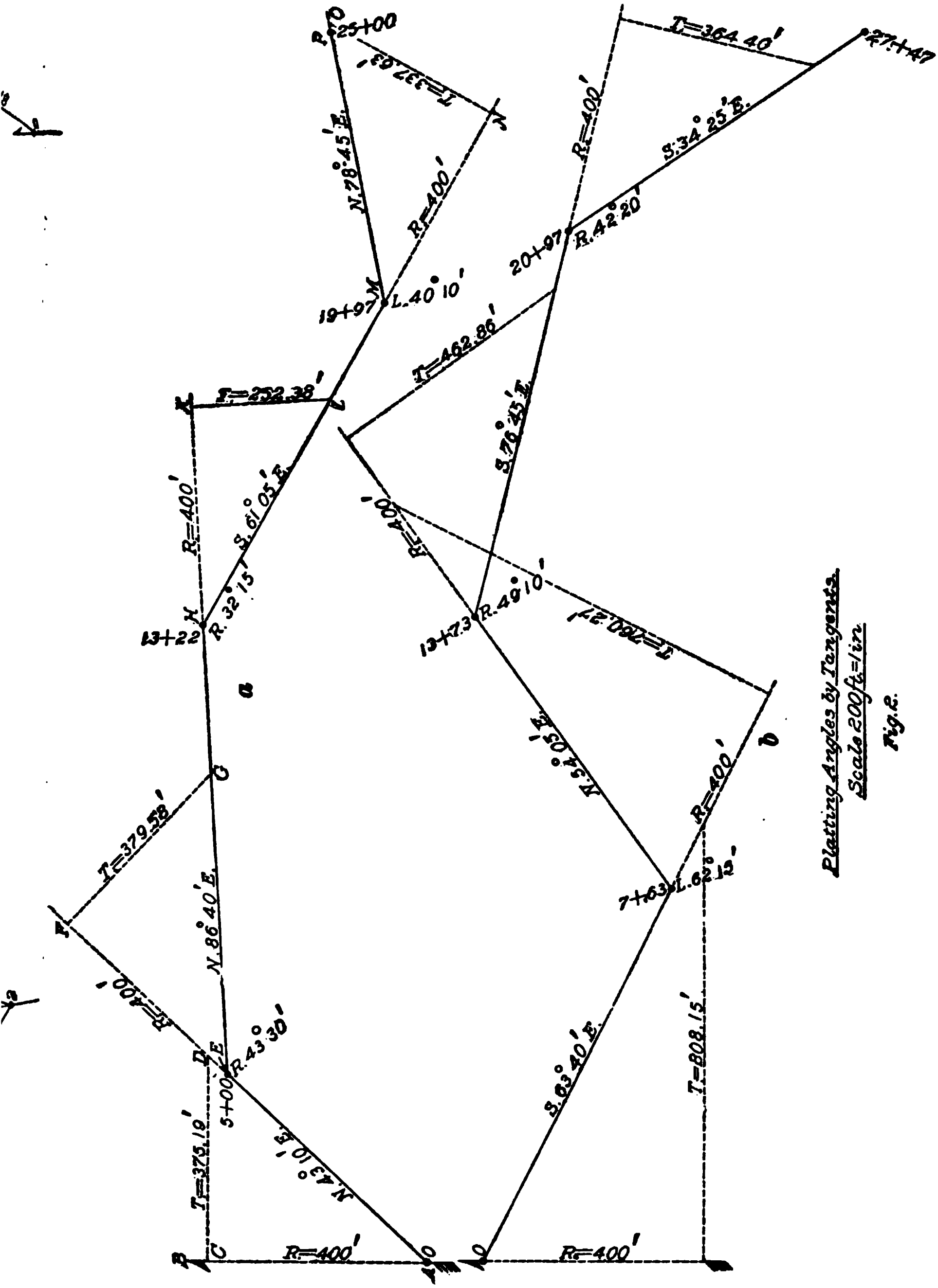

Platting Angles by Tangents.
Scale 200ft.=1in
Fig. 2.

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will draw line *a* to a scale of 200 feet to the inch, and lines *b* and *c* to a scale of 2 chains to the inch. The notes of line *a* are given below.

Let *A* be the starting point of the line, which we number Station 0. Let the arrow *N S* give the direction of the magnetic meridian. Through *A* draw a meridian *A B* parallel to *N S*. The bearing of the first course is N 10° 15′ E. From the meridian passing through *A* lay off this bearing angle with a protractor, as directed in Art. **1352.** The first course is 375 ft. Draw a line through *A* having the given bearing, and scale the distance 375 ft. This will bring us to Sta. 3 + 75, which we denote by the letter *C*, where a bearing of N 60° E is taken. The end of this course is at Sta. 6 + 90. The length of the second course will, therefore, be the difference between 6 + 90 and 3 + 75, which is 315 feet. Through *C* draw a meridian *C D*, from which lay off the bearing angle of 60° and draw a line marking the second course. Scaling the distance 315 feet we reach Sta. 6 + 90, which we call *E*. Here a bearing N 83° 30′ E is taken. Through *E* draw a meridian *E F*, and from it lay off the bearing N 83° 30′ E. The end of this course is at Sta. 10 + 40. Its length will, therefore, be the difference between 10 + 40 and 6 + 90, which is 350 ft.

NOTES FOR LINE *a*.

Stations.	Bearings.
28 + 15	End of line.
23 + 55	S 45° 00′ E
18 + 92	S 70° 45′ E
14 + 20	N 80° 30′ E
10 + 40	S 81° 20′ E
6 + 90	N 83° 30′ E
3 + 75	N 60° 00′ E
0	N 10° 15′ E

Scale off this distance from *E*, locating Sta. 10 + 40, which we call *G*. The bearing at *G* is S 81° 20′ E. Through *G* draw the meridian *G H*. As the bearing is S E, the meridian will fall below the station, from which lay off the bearing S 81° 20′ E, and draw a line in the direction of this course. The next bearing is taken at

Sta. 14 + 20. The length of the course is, therefore, the difference between 14 + 20 and 10 + 40, which is 380 ft. Call Sta. 14 + 20, *K*. Through *K* draw the meridian *K L*. The bearing here is N 80° 30′ E. From the meridian *K L*, lay off this bearing and draw a line in the direction of the course. In a similar manner locate the remaining stations and lay off the remaining bearings of the line. The bearing of each course should be distinctly written above it, the letters reading in the same direction in which the line is measured.

The notes for the lines *b* and *c* are as follows:

NOTES FOR LINE *b*.

Stations.	Bearings.	Distances.
1	N 40½° E	4.22 chains
2	N 65¼° E	6.75 chains
3	S 75½° E	8.70 chains
4	S 45¼° E	6.60 chains
5	S 20¼° W	5.18 chains

NOTES FOR LINE *c*.

Stations.	Bearings.	Distances.
1	S 47° E	6.60 chains
2	N 20¼° E	8.80 chains
3	S 80° E	4.32 chains
4	S 20° E	6.54 chains
5	N 65½° E	7.48 chains

1354. The regular 100-foot stationing is used in railroad and highway surveying, but in land surveying the lengths of the courses are given in surveyors' chains. As the fractional parts of chains are given decimally, the length of each course is readily scaled on the plat with a decimal scale. The notes of line *b* are platted as follows: The starting point is called Sta. 1, and so marked on the plat. Call Sta. 1, *A*. Through *A* draw a meridian *A B*, and from it lay off the first bearing, N 40½° E. The first course is 4.22 chains in length, which lay off to a scale of 2 chains to the inch, locating Sta. 2, which call *C*. Through *C* draw a meridian *C D*, and lay off the given bearing N 65½° E. The course with this bearing is 6.75 chains in length, which scale off, locating Sta. 3. In similar manner plat the remainder of line *b*, and also line *c*. Mark distinctly each course, giving its direction and length, being careful that the figures and letters shall read in the same direction in which the line is being run.

1355. To Lay Off an Angle by its Tangent.—In laying off an angle by its tangent, the line from which the angle is turned is prolonged to a distance equal to the length of the assumed radius. The length of the tangent of the given angle is then found in terms of the assumed radius and the tangent platted. A line joining the angular point with the extremity of the calculated tangent will give the direction of the required line, which is then measured to the given scale.

Let *A B*, in Fig. 330, be the given line, from which an angle of 30° 15′ is to be laid off to the right at the point *B*. Produce *A B* to *C*, making $B\ C = 400$ feet, the length of the assumed radius. The tangent of 30° 15′ in terms of a radius 1, is .58318, which, multiplied by 400 feet, the length of the assumed radius, gives 233.27 feet, the length of the required tangent. At *C* erect a perpendicular to *B C* 233.27 feet in length, equal to the calculated tangent. Denote the end of this tangent by *C′*. Join *B* and *C′*. The angle $C\ B\ C' = 30°\ 15'$, the given angle, and the line *B C′* is the required line.

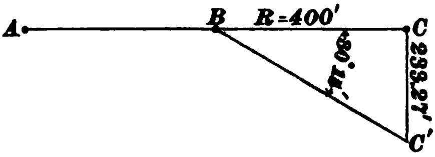


Fig. 330.

The following notes which are platted in Plate, Title: Platting Angles II, Fig. 2, the student will plat to a scale of 200 feet to the inch:

NOTES FOR LINE *a*.

Stations.	Angles.	Bearings.
25 + 00	End of Line	
19 + 97	L. 40° 10′	N 78° 45′ E
13 + 22	R. 32° 15′	S 61° 05′ E
5 + 00	R. 43° 30′	N 86° 40′ E
0		N 43° 10′ E

The notes for line *a* are platted as follows: Having adjusted the paper to the drawing board and drawn a meridian *N S*, fix the starting point *A*, which number 0. The first course is 500 feet in length, which plat by drawing a meridian *A B* through Sta. 0, and scale off 400 ft. equal to the length of the assumed radius *A C*. The bearing of the first course is N 43° 10′ E. The tangent of 43° 10′ is .93797, which, multiplied by 400, the length of the radius, gives 375.19, the length of the required tangent. Erect a perpendicular to *A B* at *C*, and on this perpendicular scale off to the right, the tangent 375.19 ft., calling the extremity of the tangent *D*. Draw *A D*. The angle *C A D* will be 43° 10′. The first course is 500 feet in length, which scale off on the line *A D* at 200 ft. to the inch, locating Sta. 5 + 00 at *E*. The angle at Sta. 5 + 00 is 43° 30′ to the right. Produce *A E* and scale off the radius *E F* = 400 ft. The tangent of 43° 30′ = .94896, which, multiplied by 400, gives 379.58 ft., the length of the required tangent. Erect a perpendicular to *E F* at *F*, and scale off, to the right, the tangent 379.58 ft., locating the point *G*. Draw *E G*. The angle *F E G* is 43° 30′, and the line *E G* the required line, the bearing of which is N 86° 40′ E. Produce *E G* to *H*, making *E H* = 1,322 − 500 = 822 ft. in length.

The line changes direction again at Sta. 13 + 22, where an angle of 32° 15′ is turned to the right. Denote Sta. 13 + 22 by *H*. Produce *E H* 400 feet, equal to the assumed radius, calling its extremity *K*. The tangent of 32° 15′ = .63095, which, multiplied by 400 feet, gives 252.38 feet, the length of the required tangent. Erect a perpendicular to *H K* at *K* and on that perpendicular scale off, to the right, the tangent 252.38 feet, locating the point *L*. Join *H* and *L*. The angle *K H L* is 32° 15′ and the bearing of *H L* is S 61° 05′ E.

The line changes direction again at Sta. 19 + 97. Call this station *M*. The angle at this point is 40° 10′ to the left. Produce *H M* 400′ to *N*, and at *N* erect a perpendicular to *M N*. The tangent of 40° 10′ is .84407, which, multiplied by 400, gives 337.63 feet, the length of the required tangent. On the perpendicular to *M N* scale off, to the left, this tangent, locating the point *O*. Join *M* and *O*. The end of the line is Sta. 25 + 00. The length of the last course is readily found by subtracting 19 + 97 from 25 + 00. The difference, 503 feet, is scaled off on *M O*, locating the point *P*, the end of the line. The bearing of *M P* is N 78° 45′ E. In a similar manner plat the notes of line *b*.

NOTES FOR LINE *b*.

Stations.	Angles.	Bearings.
27 + 47	End of Line	
20 + 97	R. 42° 20′	S 34° 25′ E
13 + 73	R. 49° 10′	S 76° 45′ E
7 + 63	L. 62° 15′	N 54° 05′ E
0		S 63° 40′ E

1356. To Lay Off an Angle by Latitude and Departure.—The subject of latitudes and departures was discussed in the section on Land Surveying, and the theory needs no explanation in connection with this subject.

Suppose the bearing of a line is N 40° E, and its length is 300 feet. Its latitude and departure are calculated as follows:

Distances.	Latitudes.	Departures.
300 ft.	2298	1928
00 ft.	0000	0000
0 ft.	0000	0000
300 ft.	+ 229.8 ft.	+ 192.8 ft.

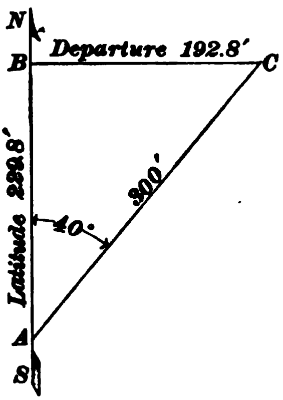


FIG. 331.

The student should bear in mind that north latitudes and east departures are +, and south latitudes and west departures are —. Let *A*, in Fig. 331, be the station at which the bearing is taken. Through *A* draw the meridian *N S*. From *A* upwards scale the calculated latitude 229.8 ft., marking the extremity *B*. At *B* erect a perpendicular to the meridian *N S*, drawing it from left to right, as the bearing is east. On this perpendicular scale off the calculated departure 192.8 ft., locating the point *C*. Join *A* and *C*. The angle *B A C* is 40°, equal to the given bearing, and *A C* is equal to the length of the given course, viz., 300 ft.

EXAMPLE.—Calculate the latitudes and departures of the following courses, and plat them by means of total latitudes and total departures from Sta. 1.

Stations.	Bearings.	Distances.	Latitudes.		Departures.	
			N +	S −	E +	W −
1	N 10¼° E	250 ft.	246 ft.		44.5 ft.	
2	N 41¾° E	123 ft.	91.76 ft.		81.92 ft.	
3	N 84½° E	215 ft.	20.64 ft.		214.03 ft.	
4	S 25¼° E	210 ft.		189.94 ft.	89.57 ft.	

Solution.—

Distances.	Latitudes.	Departures.
250 ft.		
200 ft.	1968	0356
50 ft.	4920	0890
0 ft.	0000	0000
250 ft.	246.000 ft.	44.500 ft.
123 ft.		
100 ft.	0746	0666
20 ft.	1492	1332
3 ft.	2238	1998
123 ft.	91.758 ft.	81.918 ft.
215 ft.		
200 ft.	0192	1991
10 ft.	0096	0995
5 ft.	0479	4977
215 ft.	20.639 ft.	214.027 ft.
210 ft.		
200 ft.	1809	0853
10 ft.	0904	0427
0 ft.	0000	0000
210 ft.	189.940 ft.	89.570 ft.

Stations.	Total Latitudes from Station 1.	Total Departures from Station 1.
1	0.00 ft.	0.00 ft.
2	+ 246.00 ft.	+ 44.50 ft.
3	+ 337.76 ft.	+ 126.42 ft.
4	+ 358.40 ft.	+ 340.45 ft.
5	+ 168.46 ft.	+ 430.02 ft.

The platting of the courses is as follows: On the meridian *N S*, Fig. 332, take a point which call Sta. 1. The total latitude of Sta. 2 is + 246 feet, and, as it is a plus latitude, it must be scaled off on the meridian above Sta. 1, locating the point *A*. The total departure of Sta. 2 is +44.5 feet. This departure will therefore be to the right of the meridian *NS*. At *A*, erect a perpendicular to the meridian, and upon it scale off the total departure 44.5 feet, locating Sta. 2. The line joining Stas. 1 and 2, i. e., the first course, will have a bearing of N 10¼° E. Its length, viz., 250 feet, we write on the plat, the figures reading in the same direction in which the line is being run.

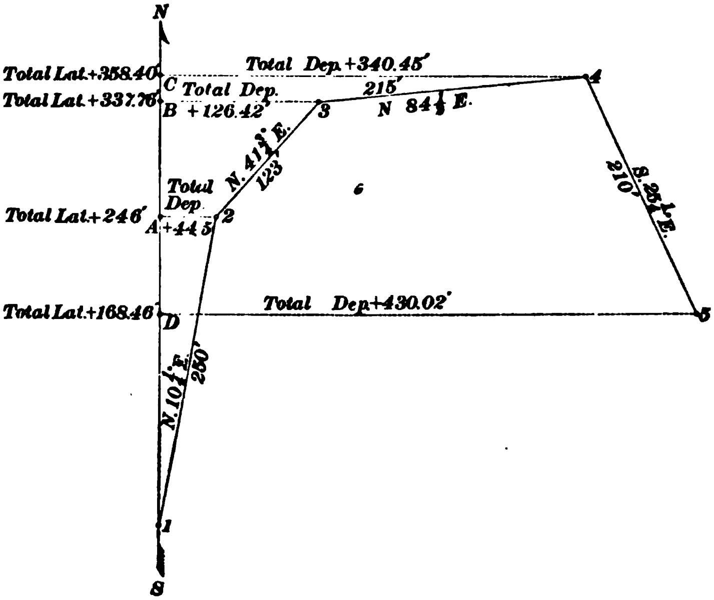


FIG. 332.

The total latitude of Sta. 3 is +337.76 feet, which we scale off on the meridian above Sta. 1, locating the point *B*, and on a perpendicular to the meridian at *B*, we scale off the total departure of Sta. 3, which is + 126.42 feet, locating Sta. 3. The line joining Stas. 2 and 3 has a bearing of N 41¾° E, and length of 123 feet. The total latitude of Sta. 4 is + 358.4 feet, which we scale off on the meridian above Sta. 1, locating the point *C*, where we erect a perpendicular to the meridian and upon it scale off the total departure of Sta. 4, viz., + 340.45 feet, locating Sta. 4. The line joining Stas. 3 and 4 will have a bearing of N 84½° E, and a length of 215 feet. The total latitude of Sta. 5 is + 168.46 feet, which we scale off on the meridian locating the point *D*, where we erect a perpendicular to the meridian and upon it scale off the total departure of Sta. 5, viz., + 430.02 ft., locating that point. The line joining Stas. 4 and 5 has a bearing of S 25¼° E, and a length of 210 ft. This method of platting bearings or angles is more accurate than either of the foregoing methods, as each course is platted independently. Great care must, however, be observed in making the additions by which total latitudes and departures are obtained. Tables of latitudes and departures are commonly calculated to quarter degrees. See table of Latitudes and Departures. Where angles are read to single minutes, a table of sines and cosines may be used to advantage. The two following formulas should be memorized:

$$\text{Latitude} = \text{distance} \times \cos \text{bearing}. \qquad \textbf{(97.)}$$

$$\text{Departure} = \text{distance} \times \sin \text{bearing}. \qquad \textbf{(98.)}$$

1357. In preliminary railroad work, angles are commonly platted by tangents, but on difficult parts of the line where all dependence must be placed on a paper location, latitudes and departures should be used and the line platted to a scale that will admit of full topographical details.

For practice in platting lines by latitudes and departures, the following examples are given. The notes of Example 1 are platted in Fig. 333, and those of Example 2 are platted in Fig. 334.

The student should carefully study the different steps given under Art. **1356,** and illustrated in Fig. 332, before undertaking to plat the following notes. He should calculate the latitudes and departures for each course, comparing his results with those given in the text and likewise with the total latitudes and departures for platting. These plats he will submit for inspection, together with Plate, Title: Platting Angles II.

EXAMPLE 1.

Stations.	Bearings.	Distances.	Latitudes.		Departures.		Total Latitudes from Station 1.	Total Departures from Station 1.
			N +	S −	E +	W −		
1	N 29¾° E	4.33 ch.	375.9 li.	....	214.9 li.	....	0.0 li.	0.0 li.
2	N 80½° E	5.17 ch.	85.3 li.	....	509.9 li.	....	+ 375.9 li.	+214.9 li.
3	N 89¼° E	4.07 ch.	5.3 li.	....	407.0 li.	....	+ 461.2 li.	+724.8 li.
4	S 10° E	6.40 ch.	....	630.3 li.	111.1 li.	....	+ 466.5 li.	+1,131.8 li.
5	S 21° W	7.11 ch.	....	663.8 li.	....	254.8 li.	− 163.8 li.	+1,242.9 li.
6	N 31¼° W	5.20 ch.	444.6 li.	....	....	269.8 li.	− 827.6 li.	+988.1 li.
7	N 51¼° W	4.45 ch.	278.6 li.	....	....	347.1 li.	− 383.0 li.	+718.3 li.
8	S 27½° W	5.85 ch.	....	518.9 li.	....	270.1 li.	− 104.4 li.	+371.2 li.
9	End of Line.		....	....	....	....	− 623.3 li.	+101.1 li.

EXAMPLE 2.

Stations.	Bearings.	Distances.	Latitudes.		Departures.		Total Latitudes from Station 1.	Total Departures from Station 1.
			N +	S −	E +	W −		
1	N 14½° W	798 ft.	772.6 ft.	....	....	199.8 ft.	....	....
2	N 89¾° W	453 ft.	1.9 ft.	....	....	453.0 ft.	+ 772.6 ft.	−199.8 ft.
3	S 49¼° W	445 ft.	....	290.5 ft.	....	337.1 ft.	+ 774.5 ft.	−652.8 ft.
4	S 60¾° E	608 ft.	....	297.1 ft.	530.5 ft.	....	+ 484.0 ft.	−989.9 ft.
5	S 70½° W	510 ft.	....	170.2 ft.	....	480.7 ft.	+ 186.9 ft.	−459.4 ft.
6	S 30½° W	387 ft.	....	333.5 ft.	....	196.5 ft.	+ 16.7 ft.	−940.1 ft.
7	S 66¼° E	648 ft.	....	260.9 ft.	593.1 ft.	....	− 316.8 ft.	−1,136.6 ft.
8	N 48° E	615 ft.	411.5 ft.	....	457.0 ft.	....	− 577.7 ft.	−543.5 ft.
9	End of Line.		....	....	....	....	− 166.2 ft.	−86.5 ft.

A piece of drawing paper one-half the size of an ordinary drawing plate will be large enough to contain plats of both lines. The total latitudes and departures in both examples are reckoned from Station 1, which in Fig. 333 is the most westerly station, and in Fig. 334 the most easterly one. Plat the courses of Example 1 to a scale of 2 chains to the inch, and those of Example 2 to a scale of 200 feet to the inch.

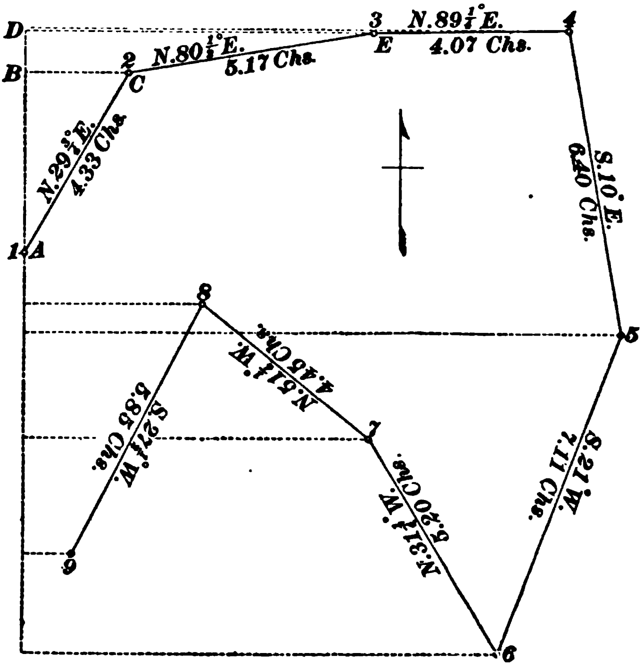


FIG. 333.

In Fig. 333, a magnetic meridian is drawn through Sta. 1, which we call *A*. We find in the column for total latitudes and departures from Sta. 1 (see Example 1), that the total latitude of Sta. 2 is + 375.9 links, which we scale off on the meridian above *A* to a scale of 2 chains or 200 links to the inch, locating the point *B*. The total departure of Sta. 2 is + 214.9 links, which we scale off at 2 chains to the inch on a perpendicular to the meridian at *B*, locating Sta. 2, which we call *C*. A line joining *A* and *C* will have the given length and bearing of the first course, viz., length 4.33 chains, and bearing N 29¾° E.

The total latitude of Sta. 3 is + 461.2 links, which we scale upon the meridian above *A* at 2 chains or 200 links to the inch, locating the point *D*. The total departure of Sta. 3 is + 724.8 links, which we scale off on a perpendicular to the meridian at *D*, locating Sta. 3, which we call *E*. A line joining *C* and *E* will have the given length and bearing of the second course, viz., length 5.17 chains, and bearing N 80½° E. In a similar manner plat the remaining courses, bearing in mind that positive latitudes are measured on the meridian above Sta. 1, and positive departures on perpendiculars to the right of the meridian, while negative latitudes

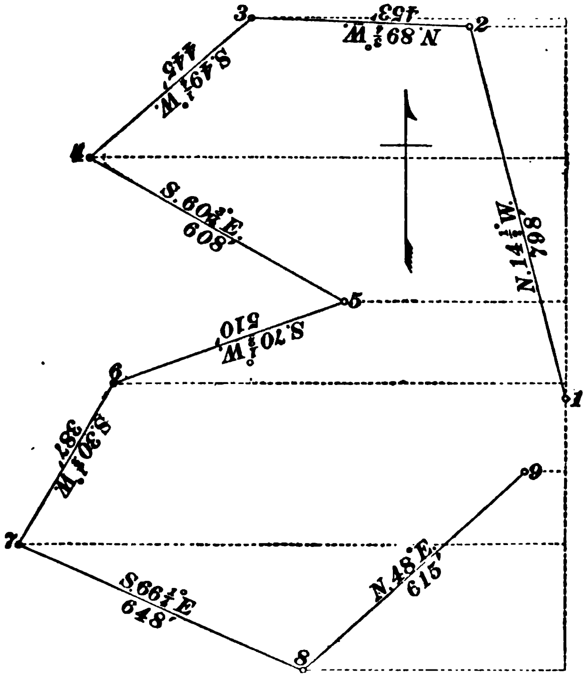


FIG. 334.

are measured on the meridian below Sta. 1, and negative departures, if there were any, on perpendiculars to the left of the meridian. The notes for Example 2 are similarly platted, excepting that the meridian passes through the most easterly station, as all the departures from Sta. 1 are negative. The lengths of the courses in this example are given in feet, and are to be platted to a scale of 200 feet to the inch. Write the bearing of each line distinctly, being careful that the letters read in the same direction in which the line is run. The student is expected to accompany each drawing with a brief description of the successive steps taken in the work.

1358. Parallel Rulers.—A parallel rule is a straight edge carried on milled rollers of equal diameter, having a common axis. They are of great service in drawing meridian lines. A magnetic meridian is drawn the entire width of the sheet which is to contain the plat. The straight edge of the rule is then made to coincide with the meridian line and then rolled across the paper until the straight edge passes through the point where the angle is to be measured. A line is then drawn following the straight edge; this will be a meridian line.

1359. The Line of Survey.—The line of **preliminary survey** is a succession of straight lines and angles, or an **angle line,** as it is commonly called, while the **located line** is a succession of straight lines and curves.

1360. Tangents and Curves.—Though this subject was considered in the section on Surveying, some additional matter may be of advantage in connection with the subject of mapping.

1361. Map of Final Location.—In mapping a final location the measurements should be made from intersection point to intersection point, and the angles platted either by tangents or by latitudes and departures. The points of curve are then located by scaling the tangent distances from the intersection points. The curve centers

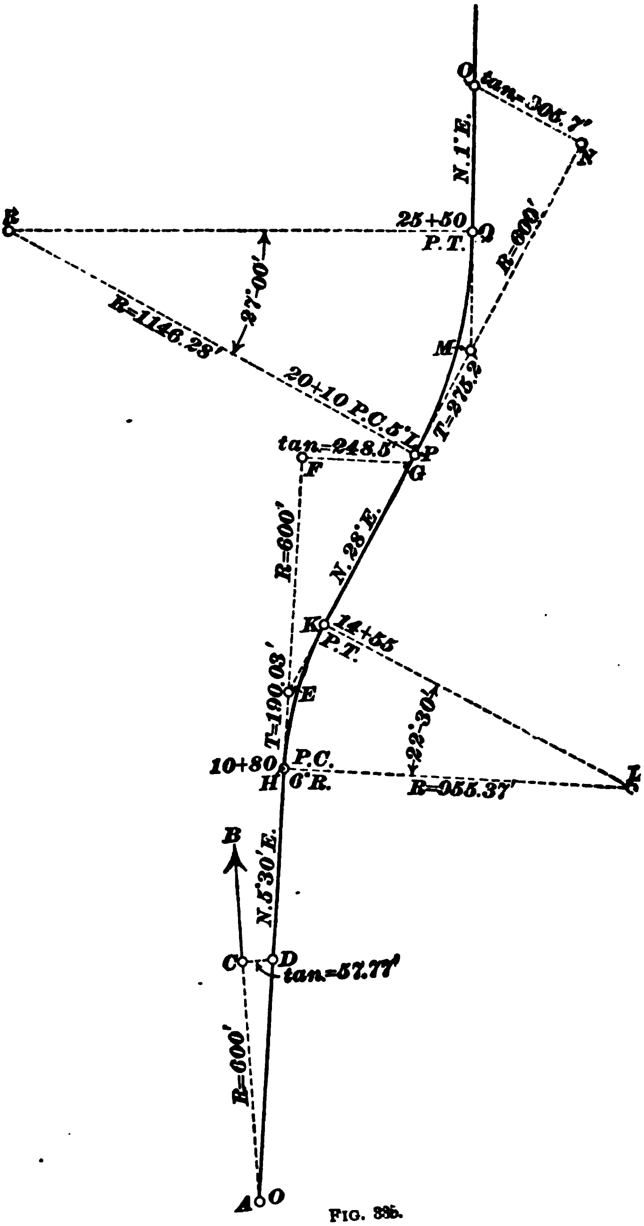

O
tan=305.7'
N
N.1°E.
R
25+50
Q
P.T.
R=600'
37°-00'
R=1146.28'
M
20+10 P.C.5°L.
T=275.2
tan=248.5'
P
F
G
R=600'
N.28°E.
K
14+55
P.T.
T=190.03'
E
22°-30'
10+80
P.C.
H
6°R.
R=955.37'
L
B
N.5°30'E.
C
D
tan=57.77'
R=600'
A
O

FIG. 335.

are best determined by describing intersecting arcs from the tangent points as centers, with radii equal to that of the given curve.

Let it be required to plat by tangents the following location notes:

Stations.	Degree of Curve.	Intersection Angle.	Tangent.	Magnetic Bearing.
25 + 50	P. T.	....	....	N 1° 00′ E
20 + 10	P. C. 5° L.	27° 00′	275.20 ft.	....
14 + 55	P. T.	....	....	N 28° 00′ E
10 + 80	P. C. 6° R.	22° 30′	190.03 ft.	....
0	....	....	....	N 5° 30′ E

The tangent distances are found by the formula

$$T = R \tan \tfrac{1}{2} I.$$

(See Art. **1251.**) The first curve is 6° R.; the intersection angle I is 22° 30′. The radius of a 6° curve is 955.37 feet. See table of Radii and Chord and Tangent Deflections.

$\frac{1}{2} I = 11° \ 15'$. Tan $11° \ 15' = .19891$; then $955.37 \times .19891$ ft. $= 190.03$ ft. $= T$, which we place in the column headed "tangent," opposite the intersection angle 22° 30′. The second curve is 5° L; the intersection angle is 27° 00′. The radius of a 5° curve is 1,146.28 feet. $\frac{1}{2} I = 13° \ 30'$. Tan $13° \ 30' = .24008$, and 1,146.28 ft. $\times$ .24008 $=$ 275.2 ft. $=$ T, which we place in tangent column opposite the intersection angle 27° 00′.

A plat of these notes is given in Fig. 335. The order of work is the following: First we select a starting point A, which we number 0, and through this point draw a meridian $A\ B$ with its north point at the top of the plat.

The first course has a bearing N 5° 30′ E. From Station 0, scale off on the meridian 600 feet, the length of our radius for platting angles. The bearing angle is 5° 30′ and its tangent .09629, which, multiplied by 600, the radius, gives 57.77 feet, the length of the required tangent. Call the extremity of the radius *C*. At *C* erect a perpendicular to *A B*, and on it lay off the tangent 57.77 feet, locating the point *D*. Join *A* and *D*, The angle $CAD = 5° 30'$. The P. C. of the first curve is at Station 10 + 80. The tangent distance, as given in the preceding table, is 190.03 feet. Hence, the distance from the starting point to the first intersection point is the sum of 1,080 and 190.03 feet, which is 1,270.03 feet. Produce *A D*, making a total distance of 1,270.03 feet to the point of intersection *E*, and 600 feet additional for the radius by which the next angle is platted. Call the extremity of this radius *F*. The intersection angle of the first curve is 22° 30'. Its tangent is .41421, which, multiplied by 600, the given radius, gives 248.52 feet as the required tangent for platting the angle. At *F* erect a perpendicular to the radius *E F* and scale off the tangent $FG = 248.52$ feet, locating the point *G*. Join *E* and *G*. The angle *F E G* is 22° 30', and the bearing of the tangent *E G* is N 28° E. Next, from the point of intersection *E*, scale off on the lines *E D* and *E G* the tangent distance 190.03 feet, locating the P. C. at *H*, Station 10 + 80, and the P. T. at *K*, Station 14 + 55. Now, from *H* and *K* as centers with a radius 955.37 feet = radius of 6° curve, describe arcs intersecting at the point *L*. Then, from *L* as a center with the same radius, describe a curve joining the points *H* and *K*. The curve *H K* will be a 6° curve and will be tangent to the lines *H D* and *K G* at the points of curve *H* and *K*. The next intersection point *M* is in the line *E G* produced. The distance between these intersection points is made up of three parts, viz., the tangent of preceding curve, which we know to be 190.03 feet; the intermediate tangent, i. e., the distance from the P. T. of the first curve to the P. C. of the second curve, and the tangent of the next curve following. The P. T. of the first curve is at Station 14 + 55; the P. C. of the second curve is at Station 20 + 10; the intermediate tangent is, therefore, the difference between 14 + 55 and 20 + 10, which is 555 feet.

The tangent of the second curve is 275.2 feet. Hence, the distance from the intersection point *E* of the first curve to the intersection point *M* of the second curve is the sum of 190.03, 555, and 275.2 ft., which is 1,020.23 ft. Produce *E G* so as to contain 1,020.23 ft., and 600 additional feet for a radius, the extremity of which call *N*. The intersection angle of the second course is 27° 00′ L., tan 27° = .50953. Radius 600 ft. × .50953 = 305.72 ft., the length of the required tangent. Accordingly, at *N* we erect a perpendicular to the radius *M N*, and on that perpendicular, scale off the tangent 305.72 feet, locating the point *O*. Join *M* and *O*. The angle *N M O* is 27° 00′, equal to the given intersection angle, and the bearing of the tangent *M O* is N 1° E. From *M* on the lines *M K* and *M O* scale off the tangent distance 275.20 feet, locating the P. C. at *P*, Sta. 20 + 10, and the P. T. at *Q*, Sta. 25 + 50. Then, from *P* and *Q* as centers with radii of 1,146.28 feet, the radius of a 5° curve, describe arcs intersecting at *R*. From *R* as a center with the same radius describe the curve *P Q*, which is a 5° curve, and is tangent to the lines *M K* and *M O* at *P* and *Q*. Write the bearing of each tangent in its proper place, being careful that the bearings shall read in the same direction in which the line is being run.

PLATE, TITLE: MAP OF RAILROAD LOCATION.

1362. This plate contains two maps of railroad location, Figs. 1 and 2, the notes for which are given in the following pages. All the angles are laid off by tangents and the notes of the *alinement* given in detail, all of which the student must carefully go over and check.

The student, before commencing these drawings, should first note that the magnetic meridian (by means of which the direction of the first tangent of each line is determined) is parallel to the right and left border lines of the plate. He must also determine by measurement from the border lines the location of the starting point 0 of each line.

Without these precautions, the lines are liable to run off the paper, necessitating a repetition of the work, and involving the erasure of lines, which always soils the paper and mars the appearance of the drawing.

He will make the drawing to a scale of 300 feet to the inch. If his scale reads only 200 feet to the inch, he will reduce the distances given to a scale of 300 feet to the inch, to their equivalent to a scale of 200 feet to the inch. The process of reduction is simple and may be readily understood from the following: A line which measures 300 feet in length to a scale of 300 feet to the inch will measure but 200 feet to a scale of 200 feet to the inch. Hence, in changing a scale from 300 feet to 200 feet to the inch the distances and dimensions will scale but $\frac{2}{3}$ of the original distances and dimensions.

EXAMPLE.—A line measures 963 feet to a scale of 300 feet to the inch. What will it measure to a scale of 200 feet to the inch?

SOLUTION.—$\frac{2}{3}$ of $963 = 642$, i.e., to a scale of 200 feet to the inch, the line will measure 642 feet.

1363. The order of platting the notes is as follows: First draw a meridian as indicated by the arrow. Next, having located the starting point A, Fig. 1, which is numbered 0, draw through that station a parallel meridian $A\,B$. We find from the notes that the direction of the back tangent $A\,A'$ (which we will consider a part of a line of railroad already constructed) is due north and south, and that Sta. 0 is the P. C. of an 8° R. curve with a central or intersection angle of 63° 10′. The tangent distance we find by the formula $T = R \tan \frac{1}{2} I$, is 440.7 feet. This distance we scale off on the meridian above the point A to a scale of 300 ft. to the inch, locating the point C, which is the intersection point of the back and forward tangents.

Next, from C on the same meridian, we scale off the radius $C\,D$ of 400 feet for laying off the angle of the first curve. The angle of this curve is 63° 10′. The tan of 63° 10′ is 1.97681. The radius 400 ft. $\times$ $1.97681 = 790.7$ ft., the length of the required tangent. At D erect a perpendicular to

NOTES FOR FIG. 1.

Station.	Deflection.	Total Angle.	Magnetic Course.	Calculated Course.
40	12° 00′			
39	6° 00′			
38 + 00	18° 00′	36° 00′		
37	12° 00′			
36	6° 00′			
35 + 00	P. C. 12° L.			
34				
33 + 04.9	10° 40.3′ P. T.	35° 50′	S 36° 30′ E	S 36° 40′ E
33	10° 30′			
32	7° 00′			
31	3° 30′			
30	7° 14.7′	14° 29.4′		
29	3° 44.7′			
28	0° 14.7′			
27 + 93	P. C. 7° R.			
24				
21				
20 + 54.9	10° 38.8′ P. T.	44° 20′	S 72° 30′ E	S 72° 30′ E
20	9° 00′			
19	6° 00′			
18	3° 00′			
17	11° 31.2′	23° 02.4′		
16	8° 31.2′			
15	5° 31.2′			
14	2° 31.2′			
13 + 16	P. C. 6° R.			
12				
11				
10				
9				
8				
7 + 89.6	15° 35′ P. T.	63° 10′	N 63° 00′ E	N 63° 10′ E
7	12° 00′			
6	8° 00′			
5	4° 00′			
4	16° 00′	32° 00′		
3	12° 00′			
2	8° 00′			
1	4° 00′			
0	P. C. 8° R.		North	North

NOTES FOR FIG. 1.

Remarks. **June 28, 1894.**

Int. Ang. = 72° 00′
12° curve, L. R. = 478.34 ft.
T. = 347.5 ft.
P. C. = 35 + 00
Length curve = 600 ft.
P. C. C. = 41 + 00
Def. 100 ft. = 6° 00′
Def. 1 ft. = 3.6′

From intersection to intersection.
Tan preceding curve = 264.8 ft.
Tan between curves = 195.1 ft.
Tan 12° curve = 347.6 ft.
Total, = 807.5 ft.
tan 72° 00′ = 3.07768
400 ft. × 3.07768 = 1,231.1 ft.

Int. Ang. = 35° 50′
7° curve, R. R. = 819.02 ft.
T. = 264.8 ft.
P. C. = 27 + 93
Length curve = 511.9 ft.
P. T. = 33 + 04.9
Def. 100 ft. = 3° 30′
Def. 1 ft. = 2.1′

From intersection to intersection.
Tan preceding curve = 389.2 ft.
Tan between curves = 738.1 ft.
Tan 7° curve = 264.8 ft.
Total, = 1,392.1 ft.
tan 35° 50′ = .72211
400 ft. × .72211 = 288.8 ft.

Int. Ang. = 44° 20′
6° curve, R. R. = 955.37 ft.
T. = 389.2 ft.
P. C. = 13 + 16
Length of curve = 738.9 ft.
P. T. = 20 + 54.9
Def. 100 ft. = 3° 00′
Def. 1 ft. = 1.8′

From intersection to intersection.
Tan preceding curve = 440.7 ft.
Tan between curves = 526.4 ft.
Tan 6° curve = 389.2 ft.
Total, = 1,356.3 ft.
tan 44° 20′ = .977
400 ft. × .977 = 390.8 ft.

Int. Ang. = 63° 10′
8° curve, R. R. = 716.78 ft.
T. = 440.7 ft.
P. C. = 0
Length of curve = 789.6 ft.
P. T. = 7 + 89.6
Def. 100 ft. = 4° 00′
Def. 1 ft. = 2.4′

Radius 1 = 400 ft.
tan 63° 10′ = 1.97681
400 ft. × 1.97681 = 790.7 ft.

NOTES FOR FIG. 2.

Station.	Deflection.	Total Angle.	Magnetic Course.	Calculated Course.
13 + 41.7	10° 15′ P. T.	32° 30′	S 79° 00′ E	S 79° 00′ E
13	9° 00′			
12	6° 00′			
11	3° 00′			
10 + 00	6° 00′	12° 00′		
9	3° 00′			
8 + 00	P. C. 6° L.			
5				
3				
0			S 46° 30′ E	

NOTES FOR FIG. 1—*Continued.*

Station.	Deflection.	Total Angle.	Magnetic Course.	Calculated Course.
69 + 10.1		End	of Line.	
61 + 65.1	15° 40′ P. T.	31° 20′	N 39° 45′ E	N 39° 40′ E
61	12° 44.1′			
60	8° 14.1′			
59	3° 44.1′			
58 + 17	P. C. 9° R.			
55				
54				
53				
52				
51				
50 + 00	17° 30′ P. T.	63° 00′	N 8° 15′ E	N 8° 20′ E
49	14° 00′			
48	10° 30′			
47	7° 00′			
46	3° 30′			
45	14° 00′	28° 00′		
44	10° 30′			
43	7° 00′			
42	3° 30′			
41 + 00	18° 00′ P. C. C. 7° L.	72° 00′	N 71° 15′ E	N 71° 20′ E

NOTES FOR FIG. 2.

Remarks.	June 28, 1894.
Int. Ang. = 32° 30′ 6° curve, L. R. = 955.37 ft. T. = 278.5 ft. P. C. = 8 + 00 Length curve = 541.7 ft. P. T. = 13 + 41.7 Def. 100 ft. = 3° 00′ Def. 1 ft. = 1.8′	Radius 1 = 400.0 ft. From Sta. 0 to P. C. = 800.0 ft. Tan 6° curve = 278.5 ft. Total from P.C. to P.I. = 1,078.5 ft. tan 32° 30′ = .63707 400 ft. × .63707 = 254.8 ft.

NOTES FOR FIG. 1—*Continued.*

Remarks.	June 28, 1894.
Int. Ang. = 31° 20′ 9° curve, R. R. = 637.27 ft. T. = 178.7 ft. P. C. = 58 + 17 Length curve = 348.1 ft. P. T. = 61 + 65.1 Def. 100 ft. = 4° 30′ Def. 1 ft. = 2.7′	From intersection to intersection. Tan preceding curve = 501.9 ft. Tan between curves = 817.0 ft. Tan 9° curve = 178.7 ft. Total, = 1,497.6 ft. tan 31° 20′ = .60881 400 ft. × .60881 = 243.5 ft.
Int. Ang. = 63° 00′ 7° curve, L. R. = 819.02 ft. T. = 501.9 ft. P. C. C. = 41 + 00 Length curve = 900 ft. P. T. = 50 + 00	From intersection to intersection. Tan preceding curve = 347.6 ft. Tan between curves = 0.0 ft. Tan 7° curve = 501.9 ft. Total, = 849.5 ft. tan 63° = 1.96261 400 ft. × 1.96261 = 785 ft.

NOTES FOR FIG. 2—*Continued.*

Station.	Deflection.	Total Angle.	Magnetic Course.	Calculated Course.
57 + 40		End	of Line.	
47 + 19	8° 46.3′ P. T.	34° 00′	N 11° 15′ E	N 11° 20′ E
47	8° 15′	....	....	....
46	5° 30′	....	....	....
45	2° 45′	....	....	....
44 + 00	8° 13.7′	16° 27.4′	....	....
43	5° 28.7′	....	....	....
42	2° 43.7′	....	....	....
41 + 00.8	14° 01.7′ P.C.C. 5° 30′ L.	52° 00′	N 45° 15′ E	N 45° 20′ E
41	14° 00′	....	....	....
40	10° 30′	....	....	....
39	7° 00′	....	....	....
38	3° 30′	....	....	....
37 + 00	11° 58.2′	23° 56.4′	....	....
36	8° 28.2′	....	....	....
35	4° 58.2′	....	....	....
34	1° 28.2′	....	....	....
33 + 58	P. C. 7° L.	....	....	....
32	....	....	....	....
30 + 36.6	13° 27.8′ P. T.	48° 10′	S 82° 30′ E	S 82° 40′ E
30	12° 00′	....	....	....
29	8° 00′	....	....	....
28	4° 00′	....	....	....
27 + 00	10° 37.2′	21° 14.4′	....	....
26	6° 37.2′	....	....	....
25	2° 37.2′	....	....	....
24 + 34.5	P. C. 8° L.	....	....	....
24	....	....	....	....
23	....	....	....	....
22 + 14.4	9° 39′ P. T.	44° 30′	S 34° 20′ E	S 34° 30′ E
22	9° 00′	....	....	....
21	4° 30′	....	....	....
20 + 00	12° 36′	25° 12′	....	....
19	8° 06′	....	....	....
18	3° 36′	....	....	....
17 + 20	P. C. 9° L.	....	....	....
17	....	....	....	....
15	....	....	....	....

NOTES FOR FIG. 2—*Continued.*

Remarks. **June 28, 1894.**

Int. Ang. = 34° 00′
5° 30′, L. R. = 1,042.14 ft.
T. = 318.6 ft.
P. C. C. = 41 + 00.8
Length curve = 618.2 ft.
P. T. = 47 + 19
Def. 100 ft. = 2° 45′
Def. 1 ft. = 1.65′

From intersection to intersection.
Tan preceding curve = 399.5 ft.
Tan between curves = 0.0 ft.
Tan 5° 30′ curve = 318.6 ft.
Total, = 718.1 ft.
tan 34° 00′ = .67451
400 ft. × .67451 = 269.8 ft.

Int. Ang. = 52° 00′
7° curve, L. R. = 819.02 ft.
T. = 399.5 ft.
P. C. = 33 + 58
Length curve = 742.8 ft.
P. C. C. = 41 + 00.8
Def. 100 ft. = 3° 30′
Def. 1 ft. = 2.1′

From intersection to intersection.
Tan preceding curve = 320.4 ft.
Tan between curves = 321.4 ft.
Tan 7° curve = 399.5 ft.
Total, = 1,041.3 ft.
tan 52° 00′ = 1.27994
400 ft. × 1.27994 = 512 ft.

Int. Ang. = 48° 10′
8° curve, L. R. = 716.78 ft.
T. = 320.4 ft.
P. C. = 24 + 34.5
Length curve = 602.1 ft.
P. T. = 30 + 36.6
Def. 100 ft. = 4° 00′
Def. 1 ft. = 2.4′

From intersection to intersection.
Tan preceding curve = 260.7 ft.
Tan between curves = 220.1 ft.
Tan 8° curve = 320.4 ft.
Total, = 801.2 ft.
tan 48° 10′ = 1.11713
400 ft. × 1.11713 = 446.8 ft.

Int. Ang. = 44° 30′
9° curve, R. R. = 637.27 ft.
T. = 260.7 ft.
P. C. = 17 + 20
Length curve = 494.4 ft.
P. T. = 22 + 14.4
Def. 100 ft. = 4° 30′
Def. 1 ft. = 2.7′

From intersection to intersection.
Tan preceding curve = 278.5 ft.
Tan between curves = 378.3 ft.
Tan 9° curve = 260.7 ft.
Total, = 917.5 ft.
tan 44° 30′ = .98270
400 ft. × .98270 = 393.1 ft.

A B, towards the right, as the curve is to the right, and upon this perpendicular scale off the calculated tangent 790.7 ft., locating the point *E*. A line joining the points *C* and *E* will give the direction of the forward tangent. On the line *C E*, scale off from *C* the tangent distance, 440.7 ft., locating the point *F*, which is the P. T. of the first curve. From *A* and *F* as centers, with radii of 716.78 ft., the radius of an 8° curve, describe arcs intersecting at *G*. Then, from *G* as a center, with the same radius, describe a curve joining the points *A* and *F*. The curve *A F* is an 8° curve and tangent to the lines *A A'* and *F E* at the points *A* and *F*.

We find from the notes that the next curve is 6° R. Its P. C. is at Sta. 13 + 16, and its central angle is 44° 20'. We find its tangent distance is 389.2 ft. We next calculate the distance from the intersection point of the first curve to the intersection point of the second curve. The distance is composed of three parts; viz., the tangent of the preceding curve, which is 440.7 ft.; the intermediate tangent, i. e., from the P. T. of the preceding curve at Sta. 7 + 89.6 to the P. C. of the second or 6° curve at Sta. 13 + 16, a distance of 526.4 ft., and the tangent of the 6° curve, which is 389.2 ft., making a total distance of 1,356.3 ft. Produce the line *C E*, and scale off from *C* on that line a total distance of 1,356.3 ft., locating the point *H*, which is the intersection point of the second or 6° curve. Produce *C H* 400 ft. to *K* for a radius in laying off the central angle, 44° 20' R., of the second curve. The tangent of 44° 20' is .977, which, multiplied by 400, gives 390.8 ft. At *K* erect a perpendicular to *H K*, and upon it scale off the tangent 390.8 ft., locating the point *L*. The line joining *H* and *L* gives the direction of the forward tangent of the second curve. Next, from the intersection point *H*, scale off on both back and forward tangents the tangent distance 389.2 ft., locating the P. C. of the second curve at *M*, Sta. 13 + 16, and its P. T. at *N*, Sta. 20 + 54.9. Next, from *M* and *N* as centers, with a radius of 955.37 ft., the radius of a 6° curve, describe arcs intersecting at *O*. Then, from *O* as a center, with the same radius, describe a curve joining the points *M* and *N*. The curve *M N* is a 6°

curve and tangent to the lines *F H* and *H L* at the points of curve *M* and *N*.

The student will draw the tangent distances and the radii and tangents for laying off angles in dotted lines, as they are simply construction lines. The line of survey he will draw in a full, bold line, as shown in the plate. The intersection points and the points of curve and tangent are marked by small circles, the latter being more fully described by their station numbers. Dotted radial lines are drawn from the center of each curve to its P. C. and P. T. On one of these radial lines the length of the radius of the curve is written, and the amount of the central angle written within the radial lines. The student will need no further directions to enable him to plat the balance of the line and also the notes for Example 2, a plat of which is given in Fig. 2.

1364. Office Curves and Beam Compass.—Office curves are curves of different radii, whose principal object

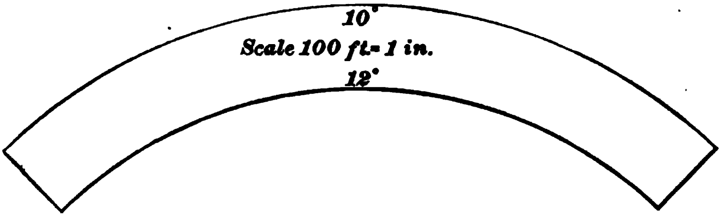


FIG. 336.

is to enable the engineer to readily select a curve which shall best fit the ground lying between tangents, as shown in the topographical map. They are commonly made of pasteboard, each piece containing arcs of two different radii, the degrees of curvature of which, together with the scale of each, being distinctly written, as shown in Fig. 336. A 10° curve to a scale of 100 feet to the inch will serve for a 5° curve to a scale of 200 feet to the inch, or a 2° 30′ curve to a scale of 400 feet to the inch. In the same way, a 12° curve to a scale of 100 feet to the inch will serve for a 6° curve to a scale of 200 feet to the inch, or a 3° curve to a scale of 400 feet to the inch. Office curves are applied directly to the contour map upon which a grade line has been platted, and the curves fitted to ground and tangent. Compound curves are as readily fitted as simple curves. A satisfactory line being decided upon, the tangent distances are calculated and the curves struck with a compass.

When the radius is of considerable length, it is difficult to describe a true circle with the ordinary compass and lengthening bar.

An accurate substitute is found in the **beam compass**, which consists of two upright legs; one pointed and fixed at the center of the circle; the other leg carrying either pencil or pen, with which the circle is described. Both legs are clamped to a horizontal arm called a beam, which is lengthened or shortened to suit the radius. A cut of a beam compass is shown in Fig. 337. *A B* is the beam, to which is clamped the needle point at *C* and the pen or pencil at *D*. At *E* is a milled-headed screw, which gives slow movement to the pen or pencil at *D* and adjusts it to the required radius.

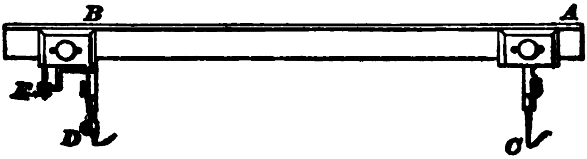


FIG. 337.

TOPOGRAPHICAL DRAWING.

1365. General Definition.—Topographical drawing consists of the representation of the different features of any portion of the earth's surface. The different features will comprise all its inequalities of surface, such as hills, hollows, streams, lakes, valleys, and plains; the location of towns, highways, canals, and railroads. Detailed topographical maps give individual dwellings, boundaries of fields, their owners, the character of the vegetation, etc.

1366. Systems.—There are three principal systems of representing topographical features, viz.:

1. By level contours or horizontal sections.
2. By lines of greatest slope perpendicular to contours.
3. By shades from vertical light.

1367. Ridge Lines and Valley Lines.—**Ridge lines** are the lines which divide the water falling upon them and from which it passes off on opposite sides. They are the lines of least slope when looking along them from above downwards, and the lines of greatest slope when looking along them from below upwards. They can be readily determined by the slope level. On these lines are found the projecting or protruding bends of the contour lines.

Valley lines are the reverse of *ridge lines*. They are indicated by the water courses which follow or occupy them. They are the lines of greatest slope when looked at from above and of least slope when looked at from below. On these lines are found all the receding or reentering points of the contour lines.

1368. Forms of Ground.—It will be found from a general examination of *any* surface that ground exists under one of the five following conditions, viz.:

1. Sloping *down* on *all sides*, i. e., a hill, as shown in Fig. 338, the direction in which water would flow being indicated by the arrows.

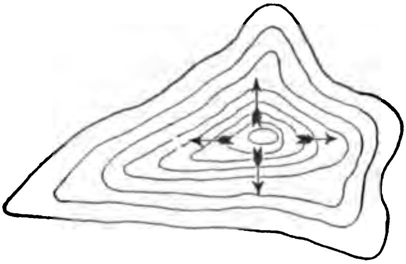


FIG. 338.

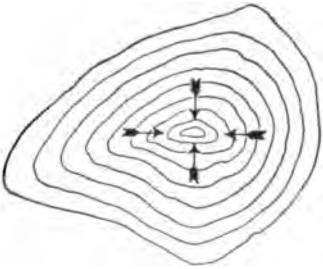


FIG. 339.

2. Sloping *up* on *all sides*, i. e., a hollow, as shown in Fig. 339.

3. Sloping *down* on *three* sides, i. e., shoulder or promontory; the end of a ridge or watershed line, as shown in Fig. 340.

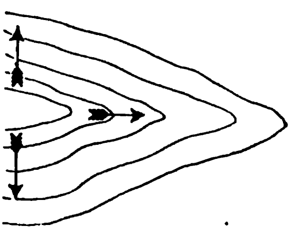


FIG. 340.

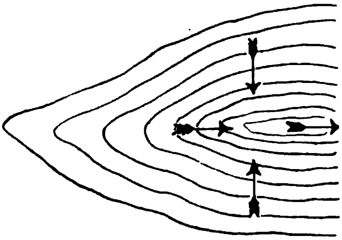


FIG. 341.

4. Sloping *up* on *three* sides and *down* on *one* side, i. e., a valley, as shown in Fig. 341.

5. Sloping *up* on *two* sides and *down* on *two* sides, alternately, called a saddle, and shown in Fig. 342.

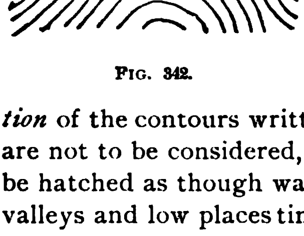


FIG. 342.

1369. Clear and Intelligible Maps.—No pains should be spared in making maps clear and intelligible. All water courses, whether occupied or dry, should be accurately sketched, and *gaps* should be left in the contour lines at suitable intervals, with the *elevation* of the contours written in them. Where time and cost are not to be considered, the lower sides of the contours may be hatched as though water were draining off them, and the valleys and low places tinted with a light shade of India ink.

Sometimes the spaces between the contour lines are tinted with India ink, increasing the tint as the depth increases.

Ground under water is commonly so represented. Beginning at low water-line, the space to the depth of 6 feet is covered with a dark shade of India ink; from 6 feet to 12 feet

with a lighter shade; from 12 feet to 18 feet with still lighter, and from 18 feet to 24 feet with lightest of all. Greater depths are marked in fathoms.

1370. Uses of Contours:

1. *To locate roads.*
2. *To obtain vertical sections—profiles.*
3. *To calculate excavation and embankment.*

In both railroad and highway location, the contour map is used in platting a grade line to which the final location should closely approximate. The paper location is then made, conforming as closely as possible to the grade line in the contour map, and from that location a final profile is platted.

When the contour map is to be used as a basis for the calculation of excavation and embankment, a hill or hollow is conceived to be divided into horizontal sections. The areas of the upper and lower bases of any section are then calculated and their average is multiplied by the altitude of the section, which gives the content of that section.

PLATE, TITLE: CONTOURS AND SLOPES.

1371. This plate contains three examples in *topographical drawing*. Each example affords practice in drawing shore lines. Fig. 1 is an example under the *first system* in which the topography is represented by *level contours*, and affords the student excellent practice in contour mapping. Figs. 2 and 3 are examples under the *second system* in which topography is represented by *lines of greatest slope or hatchings.*

First System—By Level Contours.—In Fig. 1 the situation is a steep hillside bordering upon a lake. The engineer before commencing the field work should examine the ground thoroughly in order that he may intelligently choose a method of work well suited to the situation. In the case in hand, the surface of the water in the lake is adopted as the datum plane.

It is a common and excellent practice to divide the area to be contoured into squares, the dimensions of which will depend upon the area to be treated and the degree of detail required in the work. Large areas are usually divided into squares containing 100 ft. on a side. The division lines serve as guides to those taking the levels. The intersections of the division lines being 100 ft. apart, they render the location of any intermediate point an easy task. These intersections are called stations, and are usually numbered consecutively or distinguished in some other way. In Fig. 1 the area is divided into squares 100 ft. on a side. *Base lines* $A\ X$ and $A\ H$ are first established where distant and well-defined targets may be set up and the lines carefully measured. The importance of an accurately measured base line, and of a distant fixed target can not be overestimated. The lines of division are determined by laying off lines at 90° to these bases, and are supposed to be parallel, a difficult thing to accomplish in rough country where short sights are frequent, and impossible if the initial angle of each line is not turned from the same backsight, and that a comparatively distant one. The base lines being established, the lines of division are carefully run. The vertical division lines, i. e., those parallel to general direction of the lake shore, are designated by the letters of the alphabet, the first being described as line A, the next as line B, the next as line C, and so on. The horizontal division lines are numbered consecutively, commencing with the bottom line, which is numbered 0, the next parallel to it 1, the next 2, and so on. The intersections of the division lines locate the succeeding stations on each line. This greatly simplifies the keeping of the notes, and enables the engineer to readily locate any point and briefly describe it. Thus, the starting point of line A is called line A, 0; the next intersection is called line A, 1; the next line A, 2, etc. The engineer determines the form of notes best suited to the situation. He will find a leveling rod 20 ft. in length of great assistance when working in a locality where changes in elevation are frequent and abrupt. The form of notes best adapted to the work in hand is the following, the notes being a record of the levels which are taken at each intersection of the division lines:

LEVELS OF LINE *A*.

Station.	Rod Reading.	Height of Instrument.	Elevation.
B. M.	+ 8.80	56.05	47.25
0	7.20	.....	48.80
1	13.70	.....	42.30
T. P.	− 19.72	.....	36.33
	+ 4.20	40.53	.....
2	5.70	.....	34.80
3	14.80	.....	25.70
T. P.	− 18.63	.....	21.90
	+ 3.53	25.43	.....
4	7.10	.....	18.30
5	9.30	.....	16.10
T. P.	− 18.55	.....	6.88
	+ 3.22	10.10	.....
6	9.60	.....	0.50
6 + 05	.....	.....	Edge of Lake.

1372. The **contour map** is made as follows: First we draw the outlines of the given area, 1,100 ft. in length by 750 ft. in width, to a scale of 100 ft. to the inch. These boundaries are then divided into equal spaces of 100 ft. each, as shown in the engraving, and the lines of division drawn, the boundaries being drawn full and the division lines dotted. The vertical division lines, as before stated, are designated by the letters of the alphabet, and the horizontal lines by numerals.

From the level notes we find the elevations of the stations on lines *A* and *B*. These elevations we mark on the map at their proper stations, and then locate the contour lines as follows: Beginning with line *A*, we find that the elevation of Station 0 is 48.8 ft.; that of Station 1 is 42.3 ft.; hence, the difference of elevation between these stations is $48.8 - 42.3 = 6.5$ ft. The distance between these stations is 100 ft., and the rate of fall between Stations 0 and 1 is equal to $\frac{100}{6.5} = 15.4$, which is called a descending slope of 1 to 15.4. The contours are 5 ft. apart, and, therefore, the elevations of each contour will be some multiple of 5 ft. Contour 45 ft. will come between Stations 0 and 1 of line *A*. As the elevation of Station 0 is 48.8 ft., we must, to reach contour 45, go towards Station 1 far enough to descend an amount equal to $48.8 - 45.0 = 3.8$ ft. As the rate of fall is 1 in 15.4, to fall 3.8 ft. we must go $15.4 \times 3.8 = 58.5$ ft., which brings us

LEVELS OF LINE *B*.

Station.	Rod Reading.	Height of Instrument.	Elevation.
		10.10	
7 + 12	......	......	Edge of Lake.
7	8.00	......	2.10
T. P.	− 1.24	......	8.86
	+ 19.84	28.70	......
6	8.30	......	20.40
T. P.	− 1.65	......	27.05
	+ 19.91	46.96	......
5	15.00	......	32.00
4	9.20	......	37.80
3	1.70	......	45.30
T. P.	− 2.21	......	44.75
	+ 18.88	63.63	......
2	6.90	......	56.70
T. P.	− 2.24	......	61.39
	+ 18.31	79.70	......
1	15.50	......	64.20
0	4.40	......	75.30

to contour 45. This distance we scale off to a scale of 100 ft. to the inch, marking the point where contour 45 crosses line *A* by a small dot. The next two lower contours are 40 and 35. As the elevation of Station 1 is 42.3 and that of Station 2 is 34.8, both of these contour lines will cross the line between these stations. The total fall between Stations 1 and 2 is $42.3 - 34.8 = 7.5$ ft., and the rate of fall is $\frac{100}{7.5} = 13.3$. To reach contour 40 we must fall 2.3 ft. below Station 1, and the distance will be $2.3 \times 13.3 = 30.6$ ft. To reach contour 35 we must fall 5 ft. more, and the additional distance will be $5 \times 13.3 = 66.5$ ft. We accordingly locate those contours at 30.6 ft. and at $30.6 + 66.5 = 97.1$ ft. from Station 1. The difference of elevation between Stations 2 and 3 is $34.8 - 25.7 = 9.1$ ft., and equivalent to a descending slope of 1 to 11 between them. Contour 30 will come at $4.8 \times 11 = 52.8$ ft. from Station 2. The difference of elevation between Stations 3 and 4 is $25.7 - 18.3 = 7.4$ ft., which gives a descending slope of say 1 to 14. This is not the exact rate of slope, but where decimal fractions are small and slopes easy the fractions may be ignored, as they will not to a perceptible degree affect the accuracy of the work. Contour 25 will come at $14 \times .7 = 9.8$ ft. from Station 3, and contour 20 at a point 70 ft. farther, or at say 80 ft. from Station 3. The difference of elevation between Stations 4 and 5 is $18.3 - 16.1 = 2.2$, but no contour line passes between these points. The difference of elevation between Stations 5 and 6 is $16.1 - .5 = 15.6$, which gives a slope of 1 to 6.4. This brings contour 15 at 7 ft., contour 10 at 39 ft., and contour 5 at 71 ft. from Station 5.

1373. The usual custom is to *work up the notes*, as it is called, before commencing the platting of the contours, and when a considerable portion of the ground has been covered, plat them, and thus avoid the delay incurred by frequent changes of work. Each engineer decides upon that form of notes best suited to the character of the work in hand. The following is a clear and simple form of notes in which is given the location and elevation of the contours on line *B* and on the cross-section lines between lines *A* and *B*:

Line *B*.	Contours Between Lines *A* and *B*.
Sta. 0 to 1.	from *A 0* to *B 0*.
0 + 3 to contour 75	5 ft. to contour 50
0 + 48 to contour 70	23 ft. to contour 55
0 + 93 to contour 65	42 ft. to contour 60
	61 ft. to contour 65
	80 ft. to contour 70
	99 ft. to contour 75
Sta. 1 to 2.	*A 1* to *B 1*.
1 + 56 to contour 60	12 ft. to contour 45
	34 ft. to contour 50
	57 ft. to contour 55
	79 ft. to contour 60
Sta. 2 to 3.	*A 2* to *B 2*.
2 + 15 to contour 55	1 ft. to contour 35
2 + 59 to contour 50	23 ft. to contour 40
	46 ft. to contour 45
	68 ft. to contour 50
	91 ft. to contour 55
Sta. 3 to 4.	*A 3* to *B 3*.
3 + 04 to contour 45	21 ft. to contour 30
3 + 70 to contour 40	46 ft. to contour 35
	71 ft. to contour 40
	96 ft. to contour 45

Line *B*.	Contours Between Lines *A* and *B*.
Sta. 4 to 5. 4 + 48 to contour 35	*A 4* to *B 4*. 9 ft. to contour 20 34 ft. to contour 25 59 ft. to contour 30 84 ft. to contour 35
Sta. 5 to 6. 5 + 17 to contour 30 5 + 60 to contour 25	*A 5* to *B 5*. 24 ft. to contour 20 55 ft. to contour 25 86 ft. to contour 30
Sta. 6 to 7. 6 + 02 to contour 20 6 + 29 to contour 15 6 + 56 to contour 10 6 + 83 to contour 5	*A 6* to *B 6*. 22 ft. to contour 5 47 ft. to contour 10 72 ft. to contour 15 97 ft. to contour 20
Sta. 7. 7 + 12 to contour 0, at edge of lake.	*A 7* to *B 7*. 95 ft. to contour 0, at edge of lake.

1374. The student having worked up these notes can, with a little practice, plat them rapidly, using a decimal scale. A small offset scale is very convenient in locating contours. The examples given in working up notes will enable the student to similarly treat the others. He should be systematic in his calculations and platting, completing all the calculations on one line before commencing another, and do likewise in his platting. Otherwise, confusion is sure to follow.

The elevations of the ground at the intersections of the division lines are given in the engraving. This is done for the convenience and assistance of the student. In regular office work the elevations of these points are taken from the level book, and the only elevations given in the map are those of the contours, which are written in gaps left in the contours for that purpose.

These elevations should be distinctly written, and, unless the slopes are very steep, bringing the contours very close together, the elevations should be written successively one above the other. In drawing the shore line, avoid the drawing of straight, regular lines. All shore lines, and especially those of lakes, are very irregular. A heavy line is first drawn outlining the shore, then a lighter one, at a small distance from, and parallel to, the first; then another line, at a greater distance from the second than the second is from the first; and so on until the shore line is clearly defined.

The contours themselves are to be drawn free-hand with an ordinary writing pen:

1375. Second System—By Lines of Greatest Slope.—Their direction is that which water would take in running off them. They are drawn *perpendicular* to the *contour lines*, and are called *hatchings.* An example of this system is shown in Fig. 343.

In sketching topography by this system, the topographer should hold the book directly in front of him so as to correspond with his position on the ground, drawing the lines *towards him.* If at the top of a hill, begin by drawing the lines from the bottom, and *vice versa.* To guide the hatchings, he should lightly sketch in the contour lines. Hatchings must be drawn *truly perpendicular* to the contour lines. Where the contour lines curve sharply, it is often well to draw in hatchings at considerable intervals as a guide to the main body of those drawn afterwards. Hatchings in adjoining rows should *not be continuous*, but so drawn as to *break joints.* They must not overlap, and should be drawn in *slightly wavy* lines. In drawing a hill where the slopes are steep and irregular, it is often well to draw auxiliary contours.

An example of this system is given in Fig. 2 of Plate, Title: Contours and Slopes, which represents an abrupt promontory. Its base marks the channel of a river. The ground on the opposite side of the river is generally level with occasional undulations. The degree of the slope is indicated by the spacing of the contours and the corresponding lengths and number of hatchings. The more abrupt the slope the closer together the contours and hatchings.

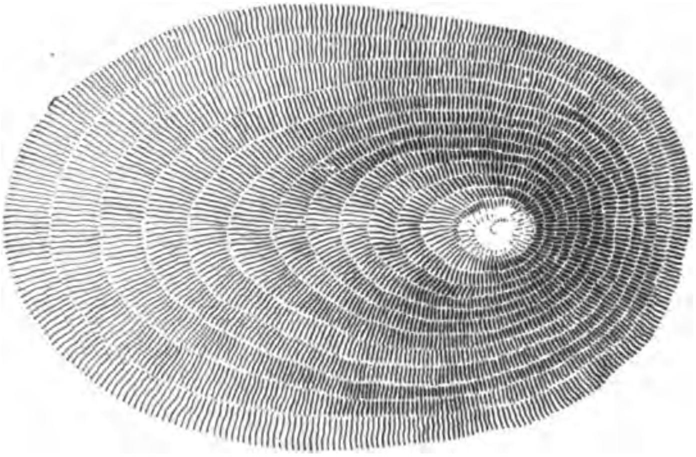


FIG. 343.

The preliminary work necessary for such a topographical map is as follows: A traverse or meander line is run, marking the windings of the stream. Having platted this meander line, the topographer takes his book containing the sketch, and from the promontory itself sketches in the main features of the surface. A hand level is of great service in determining relative elevations. From these notes the final map is made up, the work being done in the office. Fine topographical drafting should not be attempted in camp. The facilities of a well-equipped office are necessary to rapid and satisfactory work. The student is not expected to reproduce the *exact outline of the figure*, but it is expected that his work will show a proper understanding of the subject. Having drawn the outline of the river, he should draw in the contours in light pencil lines, spacing them to conform to the different slopes. It will be evident to the student that within the space represented by Fig. 2 the surface of the river at *C* and *D* will be practically the same. Hence, if the distance from the summit *A* to the river at *E* is but half the distance from *A* to *F*, the slope *A E* must be twice as abrupt as the slope *A F*. Hence, the contours which mark equal heights will be twice as far apart on the slope *A F* as on the slope *A E*. He should draw all the contours, outlining the summits at *A* and *B*, before commencing the hatchings. The short hatchings on either side of the river mark its banks. On the promontory side they are shorter than on the opposite side, as the former has the more abrupt banks. Fig. 3 of the same plate represents an irregular and abrupt sea coast. The survey for such an area would embrace a traverse of the entire shore line—of the island as well as the mainland. This traverse line should be used as a base line for auxiliary traverse lines, by means of which the summits *A*, *B*, *C*, *D*, and *E*, and any other important objects could be located. The heights of these summits could be determined either by triangulation or by the aneroid barometer. With this information as a basis, the shore line is located, the contours sketched in, and the hatchings drawn. As in the case of Fig. 2, the student is not expected to produce a *literal copy*, but to show his proficiency by furnishing a clear and finished drawing.

Hatchings should have their thickness and distance apart proportional to the steepness of the slope. The lines are made heavier as the slope is steeper, being fine for gentle slopes, and for very steep slopes the blank spaces are but half the breadth of the lines.

1376. Third System—By Shades from Vertical Light.—The steeper the slope the less light it receives. In practice, the difference in color is much exaggerated.

Various governments have prepared tables establishing the ratio of color to different slopes. The shading is applied in various ways. A rapid method, and a sufficiently accurate one for many kinds of work, is to sketch in the contours and then apply the shading in the form of India ink. Each varying tint is applied with its particular brush, care being taken not to allow any tint to dry before the succeeding tint is applied. By this means the tints are blended, giving a smooth and finished effect to the work. The tints are made light for gentle slopes and dark for steep slopes, a slope of 60° being black and one of 30° being midway between black and white, and so on.

1377. Shades by Contour Lines.—This is accomplished by interpolating additional contour lines between the regular contours. Confusion is likely to result from this method, especially where the slopes are steep, as the numerous contours are liable to run together or be confused with roads or boundaries.

CONVENTIONAL SIGNS.

1378. Sand, Rock, Etc.—Sand is represented by fine dots made by the point of a pen; gravel by coarser dots. Rocks are represented by angular, irregular masses, as would appear when seen from above and drawn in their proper places.

1379. Signs for Vegetation.—Woods are represented by scalloped circles, irregularly placed, closer or farther apart, according as the forest is dense or open. Their shadows are drawn on their lower right-hand sides, as shown in Fig. 344. Sometimes trees are drawn in elevation,

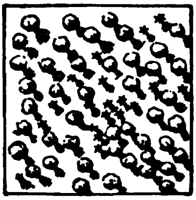


FIG. 344.

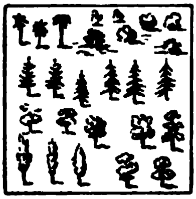


FIG. 345.

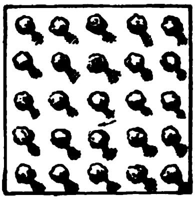


FIG. 346.

as shown in Fig. 345. This method is not admissible according to the laws of projection, but it is very effective, and deciduous trees are more readily distinguished from evergreen by this method than by that shown in Fig. 344, where deciduous trees are represented by scalloped circles and evergreens by stars.

Orchards are represented by trees in regular rows, as shown in Fig. 346.

Bushes are drawn like trees, but in smaller figures. Fig. 347 represents bushes and trees intermingled.

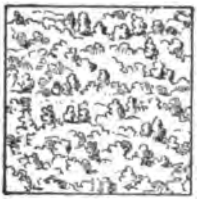


FIG. 347.

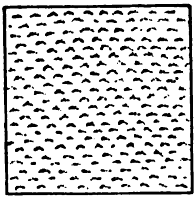


FIG. 348.

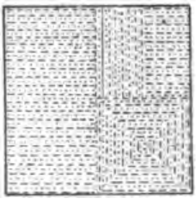


FIG. 349.

Grass land is represented by irregular groups of short diverging lines, as shown in Fig. 348.

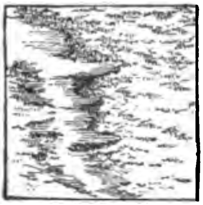


FIG. 350.

Uncultivated land is shown by interspersing the signs of sand, grass, bushes, etc.; cultivated land by parallel rows of broken lines, as in Fig. 349.

Swamp land is represented by grass, bushes, and water, as in Fig. 350.

1380. Shore Lines.—The sea-shore is represented by a line following all its windings and indentations. A short

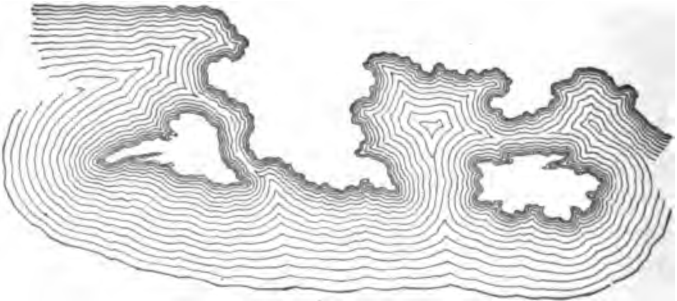


FIG. 351.

distance from the shore line, a parallel line is drawn, and a little further removed a second parallel, and so on, as in Fig. 351.

An abrupt and rocky shore is shown in Fig. 352. The irregular dotted surfaces surrounded by shore lines represent

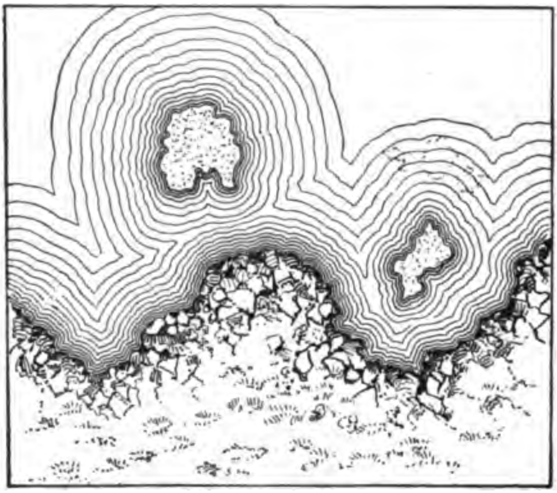


FIG. 352.

sand bars. The dotted outlines beyond the shore lines represent either shoals or sunken rocks.

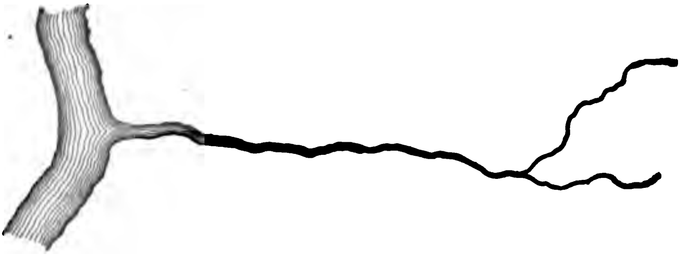


FIG. 353.

Rivers have their shore lines treated in the same manner as the sea-shore, as shown in Fig. 353.

Large brooks are represented by two parallel lines; small brooks by a single line.

1381. Grounds and Gardens.—Grounds and gardens are represented in plan as follows: The grounds, by boundaries of property and street lines; the house and other buildings, by their ground plan, and drives, walks, lawn, shrubbery, and trees, by either outlines or conventional signs. The gardens by rectangular beds and other forms of arrangement.

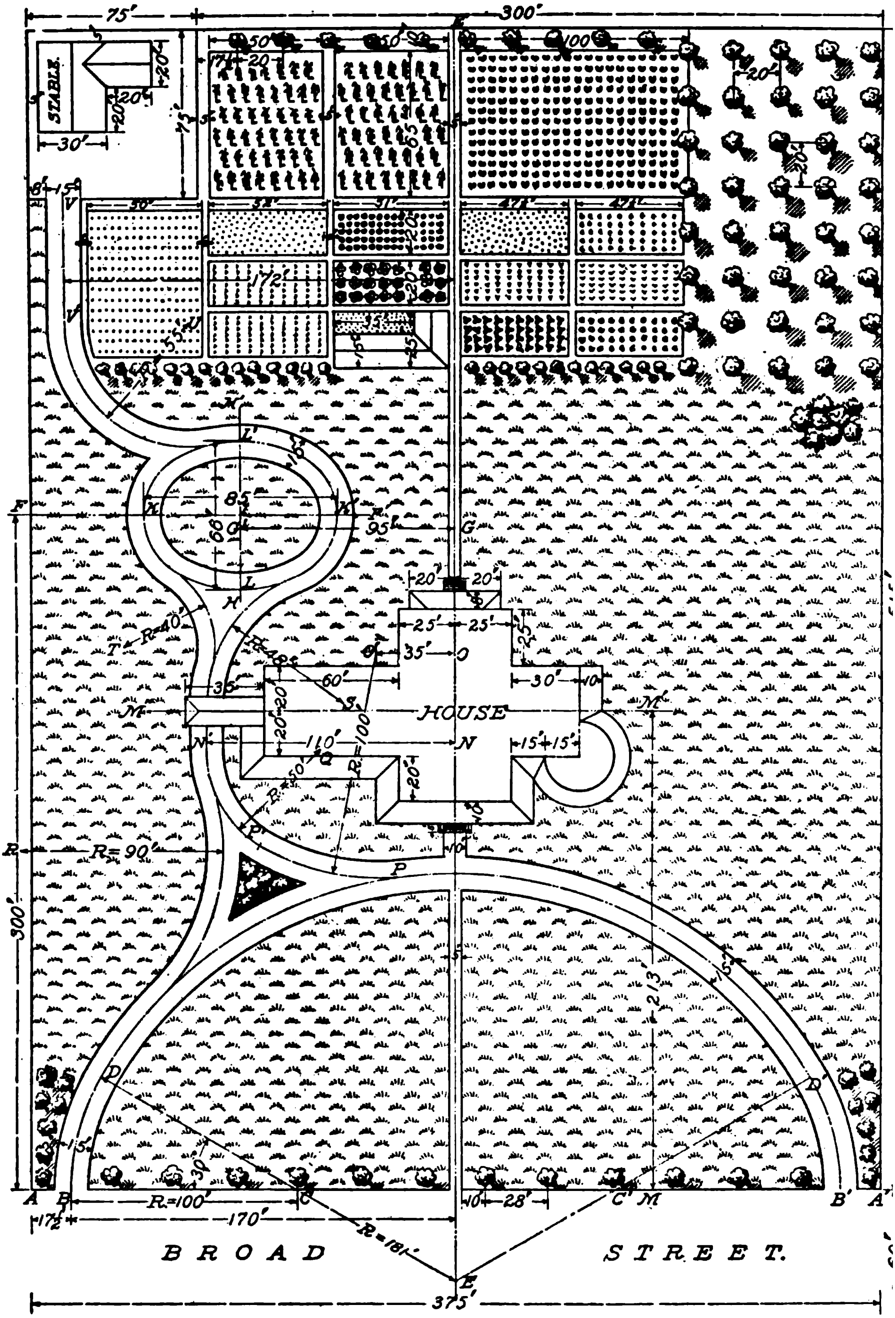
PLATE, TITLE: TOPOGRAPHICAL MAPS.

1382. Fig. 1 of this plate illustrates the use of conventional signs in practical landscape gardening, affording the student some knowledge of how a given area may be disposed so as to combine artistic arrangement with practical utility. It contains a plat of a house, grounds, and gardens drawn to a scale of 50 feet to the inch. All necessary dimensions are given for an accurate reproduction of the work. The student should first carefully study the outlines, and accurately determine their dimensions, after which the details will be a simple matter.

In drawing Fig. 1, first lay out the boundaries of the plat of ground, 375 ft. front by 515 ft. deep.

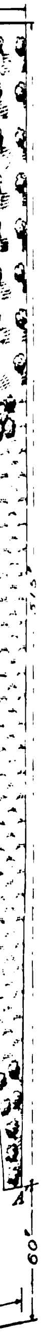
The width of the street fronting the lot is 60 ft. From the front corners of the lot A and A', measure $A\ B$ and $A'\ B'$, each 17½ ft., locating the center line of the carriage drive. From B and B' measure 100 ft., locating the centers C and C'. From these centers with equal radii $C\ B$ and $C'\ B'$ describe equal arcs $B\ D$ and $B'\ D'$ containing 30°. Produce the radii $C\ D$ and $C'\ D'$ until they intersect at E. If the work has been correctly done, E will be equidistant from the corners A and A'. Through E draw a line $E\ E'$ at right angles to the line $A\ A'$. This line will divide the lot into two equal parts and form the main base line for the location of points in the plat. With the radius $E\ D$ 181 ft. in length describe the arc $D\ D'$, completing the center line of the front carriage drive. The drive is 15 ft. in width. The inner boundary is described by simply reducing the length of the radius 7½ ft., and the outer boundary by increasing the length an equal amount. We next measure

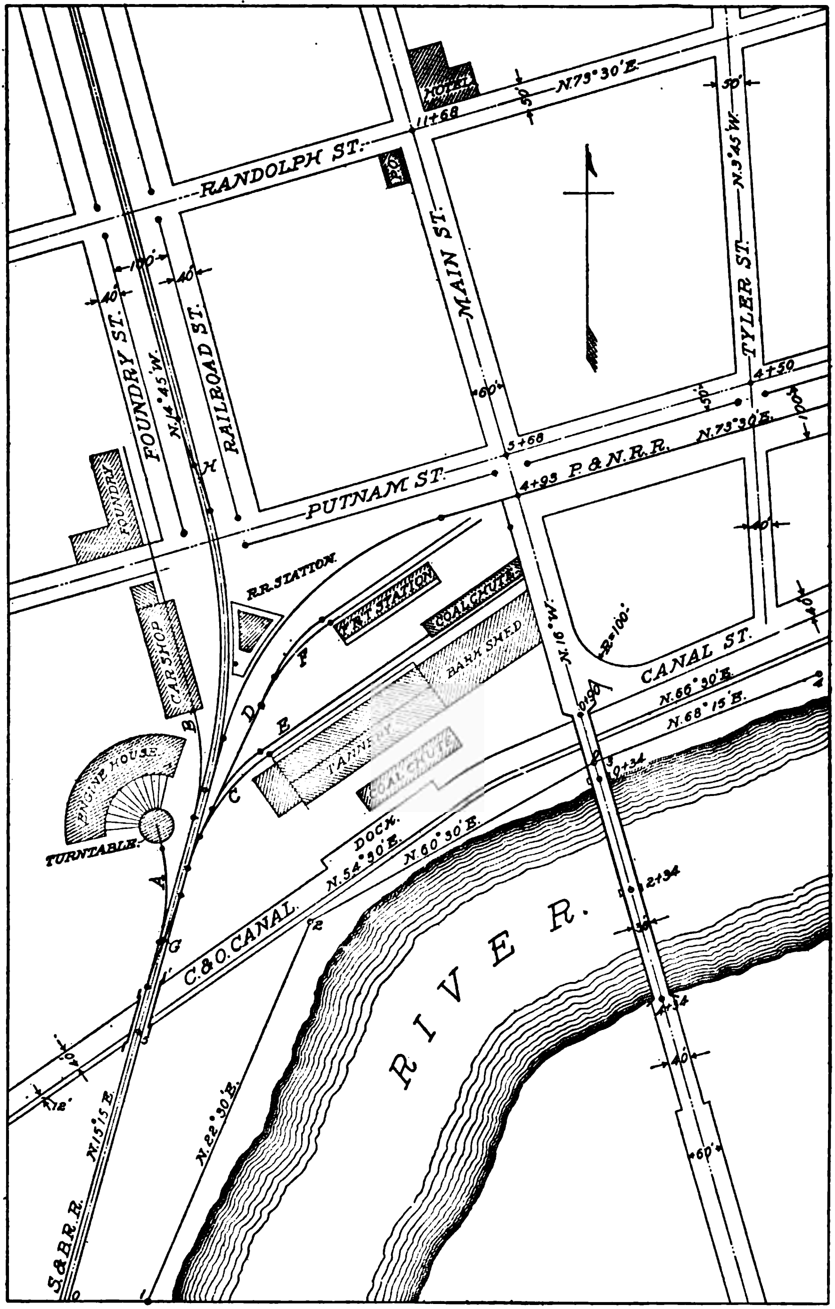
TOPOGRAP



PLAT OF HOUSE GROUNDS AND GARDENS.
Scale 50ft=1inch.
Fig. 1.

APHICAL MAPS.





PLAT OF A PORTION OF A TOWN.
Scale 200 ft = 1 inch.
Fig. 2.

e page immediately following the title page.

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from the street corner *A* on the left boundary the distance *A F* = 300 ft., and at *F* draw *F F'* perpendicular to *A F*. At some point *G* of the main base line *E E'*, draw *G G'* perpendicular to *E E'* and 95 ft. in length. At *G'*, the extremity of this perpendicular, draw a line *H H'* perpendicular to *G G'* intersecting *F F'* at *I*, which point is the center of an elliptical curve in the carriage drive. The major axis *K K'* of this ellipse is 85 ft., and the minor axis *L L'* is 66 ft. in length. This ellipse the student will draw according to the method described in the section on Geometrical Drawing. Having drawn the ellipse, the student will readily draw the border lines of the drive by increasing or reducing the lengths of the various radii.

Draw both outer and inner boundaries of the ellipse in pencil so as to form closed figures. From some point *M* of the street line draw *M M'* 213 ft. in length and parallel to *E E'*. Through *M'* draw *M' M"*, perpendicular to *E E'*. From *E E'* and *M' M"* locate the house from dimensions given. From some point *N* of *E E'*, draw *N N'* 110 ft. in length and perpendicular to *E E'*. At *N'* draw in both directions an indefinite line perpendicular to *N N'*. At some point *O* of the line *E E'* erect a perpendicular *O O'* 35 ft. in length, and through *O'* draw an indefinite line parallel to *E E'*. From a point *O'* of this line with a radius of 100 ft. describe an indefinite arc *P P'*, being careful that the extremity *P* shall be in a tangent to the center line at the carriage driveway. Then, from some point *Q*, found by trial, with a radius of 50 ft., describe an arc *P' N'* which shall be tangent to the straight line through *N'* and the arc *P P'*. In a similar manner find centers at the points *R*, *S*, *T*, and *U*, from which with the given radii describe arcs forming the center line of the carriage driveway. Having located the center line complete, the boundaries are put in as previously directed, being careful to unite the various curves with smooth, even lines. The arc described from the center *U* must be tangent to the ellipse and to the center line *V V'*, which is 15½ ft. distant from the boundary of the lot. The stable lot is 75 ft. square. The buildings may be readily located from figures given in the drawing. The kitchen garden extends from the rear boundary of the lot 150 ft. towards the street, and from the right-hand boundary to the driveway and stable yard. It is divided into rectangular plots, the various sizes being suited to the character of the vegetables grown. All dimensions necessary for a reproduction of the drawing are fully given.

1383. Fig. 2 shows a portion of a town, together with its connections with railroad and canal. The student will make the drawing to a scale of 200 feet to the inch.

Before commencing this plate the student will note that the magnetic meridian is parallel to the right and left borders of the plate. At 250 ft. from the S W corner of the plat we locate a plug, which is the starting point for the traverse of the river. Call this plug Sta. 1. Thence, run N 22° 30′ E 734 ft. to Sta. 2; thence, N 60° 30′ E 576 ft. to Sta. 3, the center of Main street; thence, N 68° 15′ E 430 ft. to Sta. 4. At 100 ft. from the starting point the river is 30 ft. to right; at Sta. 2 it is 70 ft. R.; at Sta. 3, 40 ft. R.; at Sta. 4. 44 ft. R. The river has an average width of 400 ft. The student will draw in the shore line from the offsets given above, giving it the same contour as shown in the plate. The opposite shore the student will locate by offsets, averaging 400 feet each. Again starting at Sta. 3 of the river traverse, a line is run N 16° W along the center of Main street, producing the line backwards from Sta. 3 across the river and beyond to the boundary of the plat. The several lines of survey will be used as base lines for the location of streets, railroad tracks, canal, and all objects included in the map. The starting point of each line of survey is numbered zero, and all lineal base-line measurements are referred to the starting or zero points. Distances measured on the base line are expressed in stations of 100 ft. each, and offsets are given in full. Calling Sta. 3 of the river traverse 0, we measure northward to Sta. 0 + 90, the north end of the river and canal bridge. The wing walls of the abutments diverge at an angle of 30°. At Sta. 4 + 93 is the center of the track of the P. & N. R. R., the bearing of which is N 73° 30′ E.

At Sta. 5 + 68 is the center of Putnam street, the bearing of which is N 73° 30′ E. At Station 11 + 68 is the center of Randolph street, the bearing of which is N 73° 30′ E. Produce the center lines of both Putnam and Randolph streets until they intersect the boundaries of the plat. Main street is 60 feet in width, and Putnam and Randolph streets are each 50 feet in width. Draw parallels to the center lines of these streets, locating the proper boundaries as shown in the plat. From the intersection of the center line of Main with that of Putnam street, measure eastward on the center line of Putnam street 450 ft. to the center of Tyler street, the bearing of which is N 3° 45′ W. Produce the center line of Tyler street northward until it intersects the north boundary of the plat. On the N E corner of Main and Randolph streets is a hotel fronting 110 ft. on Main and 100 feet on Randolph streets. The hotel has a depth of 60 feet from each frontage. On the S W corner of Main and Randolph streets is the postoffice, fronting 60 ft. on Main street and 40 ft. on Randolph street. Returning to the starting point, viz., Sta. 3 of the river traverse, and running southward on the center line of Main street, at Sta. 0 + 34 is the center of the shore pier, 10 ft. by 40 ft.; at Sta. 2 + 34 is the center of the channel pier, 12 ft. by 40 feet, and at Sta. 4 + 34 is the south end of the bridge. The bridge is 30 feet wide. The wing walls of the south abutment also diverge at an angle of 30°. The south approach is 40 ft. in width and 200 ft. in length. Beyond the approach the street has the full width of 60 feet.

At 100 ft. east of the S W corner of the plat is the center line of the S. & B. R. R., which has a double track, the track centers being spaced 13 ft. apart. The bearing of this tangent is N 15° 15′ E, and the given point is numbered Sta. 0. At Sta. 5 is the center of the towing path of the C. & O. canal, the bearing of which is N 54° 30′ E. At Sta. 9 + 44.1 is the P. C. of a 6° curve L. for 30°. The student will produce the center line to Sta. 12, which is the point of intersection for the curve, laying off the tangent distance of 255.9 ft. Draw the center line of each track, spacing them 6.5 ft. from the main center line. Having located the center of the 6° curve, the parallel curves are struck with a compass, increasing the radius 6.5 ft. for the outer curve and diminishing it the same amount for the inner curve. North of Putnam St. the right of way of the S. & B. R. R. is 100 ft., 50 ft. each side of the main center line. Parallel to this center line on each side of the railroad is a street 40 ft. in width. Produce the center line of the east track of the S. & B. R. R. until it intersects the center line at the P. & N. R. R. The intersection angle is 58° 15′. Unite these tangents by a 10° curve. At Sta. 6 + 63 of the S. & B. R. R. west track, is the P. C. of a 16° curve L. for 27°, which we call track *A*.

At 40 ft. from the P. T. of this curve is the center of a turntable, the diameter of which is 60 ft. From the circumference of the turntable to the inner wall of the roundhouse is 60 ft. The right-hand end wall of the roundhouse faces on a radial line from the center of the turntable, which line makes an angle of 30° 30′ with the tangent of the turnout curve. The left-hand end wall is similarly situated, making an angle of 91° 30′ with the tangent of the turnout curve. The depth of the stalls is 70 ft. The tracks at the inside wall line are spaced 15-ft. centers. From the center of the outside track to outside of end walls is 10 ft. At Sta. 8 + 92 of the west track is the P. C. of a 16° turnout curve, to left, for 30° 10′. This we call track *B*. The P. T. is on the line of the south end wall of a car shop. The side walls of the car shop are parallel to the tangent of the curve. The east wall is 15 ft. from the center line of the tangent and the west wall 50 ft., giving a total width of 65 ft. The length of the car shop is 200 ft. At the northwest corner of the car shop is an engine and boiler house 40 ft. square.

On the north side of Putnam street is a foundry, fronting 120 ft. on Putnam street and 175 ft. on Foundry street, the building having a depth of 50 ft. from both frontages. The east side of the building is parallel to the tangent of the turnout curve leading to the car shop, and 10 ft. from it. At Sta. 11 + 55, as measured on the produced tangent of the S. & B. R. R. is the south end of the platform of the R. R. station. At Sta. 12 is the south end of the station proper. The station is 67 ft. long, as measured on this tangent. The outer edge of the platform is 8 ft. from the center line of the adjacent tracks. The platform is 10 ft. wide and extends on all sides of the station, the curved walls of which are parallel to the railroad tracks.

At Sta. 8 of the S. & B. R. R. is the P. C. of a 16° turnout curve R., which leads from the east track, containing 39° 50′, and called track *C*. A tannery, 300 ft. in length by 90 ft. in width, extends from the P. T. of this curve, parallel to and 20 ft. to the right of the center line of the tangent. A bark shed of the full width of the tannery forms a continuation of the tannery and extends to Main street. A platform 10 ft. in width extends the entire length of the tannery, between it and the tracks. At the southwest corner of the tannery is an engine and boiler house 50 by 80 ft. At 70 ft. from the P. C. of track *C* and tangent to that curve, track *D* commences.

At 52.72 ft. from the commencement of track *D* is the P. C. of a 23° curve R. for 28° 38′, and called track *E*, which terminates in a tangent parallel to the tangent of track *C*, and spaced 12 ft. from it. At 364 ft. from the P. T. of track *E* is the west end of a coal chute. The south side of the chute is 8 ft. from the center of the track; the north side is 22 ft. from the center of the track, which gives the building a total width of 30 ft. It extends in length to Main street. At 257 ft. from the commencement of track *D* is the P. C. of a 16° curve R. for 31°, and called track *F*. At 8 ft. to the right and parallel to the tangent of this curve is a freight depot, extending from the P. T. of the curve and 200 ft. long by 35 ft. wide. On the south side the freight station is a platform 10 ft. wide, extending its entire length. At 312 ft. from the commencement of track *D* is the P. C. of a 23° curve R. for 31°, the tangent of which is parallel to track *F*, their center lines being spaced 12 ft.

Between Sta. 5 + 78 of the S. & B. R. R. west track, and Sta. 7 + 51, east track, is a cross-over track, which we call track *G*. This consists of two turnout curves, one commencing at each of the given stations and intersecting on the main center line. The curve commencing at Sta. 5 + 78 is a 9° 30′ curve R.; the curve commencing at Sta. 7 + 51 is also a 9° 30′ curve R., but described in the opposite direction. If carefully drawn, these curves will intersect on the main center line. Between Sta. 14 + 44.1, east track, and Sta. 16 + 17.1, west track, is another cross-over, which we call track *H*. The curves are both 9° 30′, described in opposite directions, as in the preceding case.

Starting at Sta. 5 of the S. & B. R. R., which is directly over the center line of the towpath of the C. & O. canal, the bearing of which is N 54° 30′ E, we measure northeastward along the center line of the towpath 706 ft. to the P. C. of a 12° R. for 12°. Produce the back and forward tangents of this curve until they intersect the borders of the plat. The towpath is 12 ft. wide and the canal 40 ft. wide. The canal makes the same curve as the towpath, the arcs being struck from the common center. The student will draw the boundaries of the canal and towpath in ink, but need not ink in the center line of the towpath. The north abutment of the Main St. canal and river bridge is 4 ft. from the boundary of the canal. At 443 ft. from our starting point on the towpath is the west end of a canal dock, which consists of a widening of the canal in which boats are moored while their cargoes are being discharged. This dock is 240 ft. long and 20 ft. wide, providing berths for four boats. At 120 feet from the west end of the dock is the west end of a coal chute, the south side of which is parallel to and 20 ft. from the dock. The coal chute is 40 by 200 ft. The railroad bridge over the canal is 30 ft. wide. The south abutment is placed 12 ft. and the north abutment 6 ft. from the canal, both abutments being parallel to the canal. The outer faces of the wing walls of these abutments are parallel to and 10 ft. from the center line of the nearest track.

PLATE, TITLE: MAP OF A VILLAGE.

1384. This plate represents a topographical map of a village. In making a survey of this description the engineer will select for a starting point some well-defined landmark; but as there are a score of points to choose from, the choice will depend upon the judgment of the engineer. The intersection of the center lines of two highways or the head block of a railroad switch are excellent points from which to commence a survey. The center lines of highways and railroads are the base lines from which the minor details, such as houses and other buildings are located. The quickest and best method of locating a building is to set a temporary plug on the base line near the building. Set up the transit at this point and measure the angles between the base line and two consecutive angles of the building, measuring the distances from the plug to the angles of the building. These angles and distances will locate one side of the building. A small free-hand sketch is then made, giving the base line, the station of the plug, or its distance from some known point, and the angles and distances to the side of the building. The remaining sides of the building are added to the sketch and their several lengths measured in consecutive order and marked on the sketch. These notes are quickly made and as quickly platted.

Sketches are of the greatest value in taking topographical notes. They can be made in less than half the time required for a full description, and are always more intelligible to the draftsman. Each surveyor has his individual methods, both in order of work and form of notes, and often one will consume twice as much time as another in performing the same work; but expedition is of no value if had at the cost of accuracy.

In this map all the conventional signs suited to an area of such magnitude are employed. The student will draw the map to a scale of 200 feet to the inch. The magnetic meridian is parallel to the right and left borders of the plate, the north point being at the top of the map. The center lines of the highways are given, together with their magnetic bearings, widths, and distances between the angles made by the center line. The center line of the railroad is represented by a heavy, full line, and the boundaries of the right of way, which is 100 feet in width, are represented by light full lines. The magnetic bearings of the tangents are given and the stations of the points of curve and tangent from which the lengths of the tangents are found by subtracting the station of the P. T. of one curve from the P. C. of the succeeding curve. The curves, instead of being run in from intersections of tangents, are platted as follows:

The location of the first tangent in the map being given by references to the boundary lines of the map, and the P. C. of the first curve denoted by its station, a perpendicular is erected to the given tangent, and upon it the length of the radius of the required curve is laid off to the given scale. Then, from the given center the curve is struck with a compass, being careful that the arc so struck shall contain as many degrees as the central angle of the curve, the central angle of the curve being laid off with a protractor. The intersection of the second radius with the arc will be the P. T. of the following tangent. A perpendicular is then erected to the second radius and tangent to the arc at the given intersection. The P. C. of the next curve being given, its length is readily found by subtraction. The borders of the lake are located by offsets.

The student is not expected to exactly duplicate the topography, but give the same general effect. All boundaries whose magnetic bearings are given and the location of all buildings he must faithfully repeat in his drawing.

1385. Starting at the northwest corner *K* of the map we measure eastward along the north boundary 700 ft. to *L*, the center of the S. & L. R. R., the bearing of which at that point is S 7° 45′ E. At 310 feet from this point is the P. C. of an 8° curve R., which we call Sta. $40 + 60$. The central angle of this curve is 25°, and its length 312.5 ft. The station of the P. T. we find by adding the length of the curve to the station of the P. C., giving 43 + 72.5 for the P. T. of the curve. The bearing of the forward tangent is S 17° 15′ W and its length 400 ft., making the station of the next P. C. 47 + 72.5. This curve is 9° R. and its central angle is 15°. Its length is, therefore, 166.7 ft., and the station of the P. T. 49 + 39.2. The bearing of the forward tangent is S 32° 15′ W, found by adding the central angle of 15° to the bearing of the preceding tangent. The length of the forward tangent is 610 ft. to Sta. 55 + 49.2, which is the P. C. of a 6° curve *L*. The curve is continued to the south boundary of the map.

The switch extending to the ice houses along the shore of the lake is located as follows:

Scale off from the northwest corner *K* of the map, along the north boundary 790 ft., eastward to the point *M*. From this point as a center, lay off with a protractor to the right of the north boundary an angle of 32° 30′. The bearing of this line is S 57° 30′ E. Scale 47 ft. on this line from the point in the boundary where the angle is turned. This will locate the P. C. of a 16° curve R., the station of which is 3 + 04.7. The central angle of this curve is 62° 05′, and the station at the P. T. 6 + 92.7. The bearing of the forward tangent is S 4° 35′ W. The next curve is 10° R. Its P. C. is at Sta. 10 + 93.7, and its central angle is 9° 40′, which brings the P. T. at Sta. 11 + 89.4. The bearing of the forward tangent is S 14° 15′ W. The next curve is also 10° R. Its P. C. is at Sta. 15 + 49.4, and its central angle 27° 30′, which brings the P. T. at Sta. 18 + 24.4. The bearing of the forward tangent is S 41° 45′ W. The last curve is 10° L. Its P. C. is at Sta. 24 + 24.4, and its central angle is 27° 10′, which brings the end of the line at Sta. 26 + 96.1.

With a protractor lay off from the P. C. of the 16° curve of the ice switch, a *central angle* of 18°. Draw a radius including this angle and at its extremity draw a tangent to the curve, with bearing of S 39° 30′ E. This point of tangent is the starting point of a switch leading to a coal chute. This point we call Sta. 0. At Sta. 0 + 65 of this track is the P. C. of an 18° curve R. for 36°, bringing the P. T. at Sta.

2 + 65. At this point the track enters a coal chute 50 ft. long by 30 ft. wide. The center line of the track is parallel to the sides of the chute and spaced 7 ft. from the west side. At 120 ft. from the starting point of the S. & L. R. R. is the north end of platform of a railroad station, and at 210 ft. from that point is the south end of the platform. The edge of this platform is spaced 6 ft. from the center line of the railroad track. The platform is 46 ft. in width at the north end and 50 ft. in width at the south end. The center line of a road 40 ft. wide commences at the middle point of the south end of the railroad station platform, and extends in the direction S 48° 30′ E 500 ft., where it intersects with the center line of the Scranton turnpike, the bearing of which is S 21° 30′ W. Call this point of intersection *A*.

Starting from intersection *A*, the traverse of the center line of the turnpike is as follows: S 21° 30′ W 310 ft.; thence, S 58° 30′ E 80 ft. to the west end of a bridge 20 ft. wide; thence, by same course 40 ft. to east end of bridge; thence, by same course 80 ft. to an intersection with center line of Andrews lane. Call this point of intersection *B*. Thence, S 11° 30′ W 300 ft. to intersection with center line of Waverly road. Call this point of intersection *C*. Thence, by same course 300 ft.; thence, S 8° 30′ E 250 ft.; thence, S 27° E 345 ft. to north end of a stone bridge 20 ft. wide; thence, by same course 30 ft. to south end of stone bridge; thence, by same course 125 ft. to an intersection with the center line of Newton road, the direction of which is N 63° E and width 40 ft. Call this intersection point *D*. From intersection *D* the turnpike extends in the direction N 83° E 400 ft., thence, S 76° E 325 ft.; thence, S 46° E to the south boundary of the map. The width of this turnpike is 50 ft.

Next we measure from the northwest corner *K* of the map, southward along the west boundary 344 ft. to the center *N* of the Benton road. From thence we measure along the center line of the road, N 80° 15′ E 356 ft.; thence, S 69° 45′ E 350 ft.; thence, S 89° 45′ E 45 ft. to west end of a culvert 20 ft. in width; thence, by same course, 50 ft. to east end of culvert; thence, by same course, crossing the ice switch and road leading to the railroad station and continuing to an intersection with the Scranton turnpike, which is the terminus of the Benton road. Call this point of intersection *E*.

Starting at intersection *C*, we follow the center line of the Waverly road as follows: S 55° 15′ E 197 ft. to its intersection with the center line of Lenox lane. This point of intersection we call *F*. Thence, N 74° 45′ E 500 ft.; thence, S 85° 15′ E, intersecting the old Scranton and Montrose turnpike and extending by the same course to the east boundary of the map.

Returning to the point *F*, the intersection of the center line of the Waverly road with Lenox lane, we prolong the line *C F* from *F*, a distance of 290 ft. This forms the center line of Lenox lane, and intersects with the center line of Henderson lane. These intersecting center lines form an angle of 90° with each other, making the course of Henderson lane N 34° 45′ E. Produce the center line of this lane in both directions, intersecting on the south with the Scranton turnpike, and on the north with the Waverly road. The widths of the Waverly road and Lenox and Henderson lanes are each 40 ft. Commencing at the point *B*, where the Scranton turnpike intersects the center line of Andrews lane, we follow the center line of that lane in the direction N 31° 30′ E 300 ft. to an intersection with the center line of a private lane leading in the direction S 85° 30′ E 200 ft., where it turns at right angles, 40 ft. to the right and 75 ft. to the left. Continuing along the center line of Andrews lane by the same course 100 ft., we change the direction, running N 11° 30′ E, intersecting with Hall road. The width of Andrews lane is 30 ft.

Starting from the southeast corner *O* of the map, we measure westward along the south boundary 320 ft. to the center *P* of the old Scranton and Montrose turnpike. From this point we follow the center line of the turnpike as follows: N 27° 45′ W 440 ft.; thence, N 7° 45′ W 330 ft.; thence, N 2° 15′ E 1,280 ft., intersecting the Waverly and Hall roads. The latter intersection makes the eastern limit of Hall road. Thence, N 5° 45′ W to the north boundary of the map. The width of this turnpike is 50 ft. Returning to point *E*, the terminus of the Benton and Hall roads, we follow the center line of Hall road S 89° 45′ E 165 ft. to the west end of a bridge 20 ft. wide; thence, by same course, 30 ft. to east end of bridge; thence, by same course, 400 ft. to an intersection with the center line of Prospect road, which extends in the direction N 40° 15′ E to the north boundary of the map. Call this point of intersection *G*. From the point *G* the center line of Hall road extends in the same direction, viz., S 89° 45′ E to its intersection with the center line of the old Scranton and Montrose turnpike. Call this intersection point *H*. The widths of the Hall and Prospect roads are 40 feet.

The right of way of the S. & L. R. R. is 100 ft. in width, 50 ft. each side of the center line, excepting at the station, where the railroad company's property extends in width 100 ft. on the east side from the center line of the track, and in length from the north boundary of the map to the highway. On the west it extends in width 200 ft. from the center line of the track and in length from the north boundary of the map to the Benton road. At a point 390 feet from Sta. 43 + 72.5 of the main track, as measured on the west right of way boundary, is the end of the boundary line between lands of James Henderson and John Andrews, the bearing of which is S 57° 45′ E. This boundary extends to the west boundary of the map. At a point in the center of Hall road, 10 feet west of intersection *G*, is a corner at the schoolhouse lot which fronts 100 feet on Hall road, and 220 feet in depth, as measured from the center line of the road, the sides being at right angles to the center line of the road. *All property lines bounding on or intersecting highways follow or extend to the center line of the highway.* Immediately adjoining the schoolhouse lot on the west is the lot of John Stark, with front of 200 ft. and depth of 220 ft. A point 300 ft. east of intersection *G*, as measured on the center line of Hall road, is a corner of lot of F. Swartz. This boundary is at right angles to the center line of Hall road and extends to the center of Prospect road. The other boundaries of the lot are marked by the center lines of the roads upon which the lot fronts. At a point 300 ft. south of intersection *H*, as measured on the center line of the old turnpike, is a corner of lot belonging to John Edwards. The sides of the lot are at right angles to the center line of the turnpike and the ends parallel. The lot fronts 300 ft. on the turnpike and has a depth of 425 ft. The south boundary of this lot forms a part of the north boundary of a lot belonging to Jane Gregory. A line joining the southwest corner of John Edwards's lot with the northeast corner of Henry Watson's lot completes the north boundary of the Gregory lot. The bearing of this line is N 87° 10′ E. The west boundary has a bearing of N 4° 45′ E, and extends to the center line of Waverly road, with which it forms an angle of 90°. The south and east boundaries of the lot are formed by the center line of the Waverly road and the old turnpike, the courses of which are already given. The point of intersection of the center line of the Waverly road with the old turnpike is a corner of lot belonging to A. Atherton. The north boundary extends along the center line of Waverly road 425 feet; thence, at right angles to that road, S 4° 45′ W 420 ft.; thence, on a line parallel to the Waverly road, to the center of the old turnpike. The west boundary is formed by the center line of the turnpike.

The west boundary of lot belonging to Jane Gregory forms the east boundary of lot belonging to Henry Watson, which has a frontage of 150 feet and a depth of 220 feet. The west boundary of Henry Watson's lot forms the east boundary of lot belonging to James Lenox, and extends N 4° 45′ E 220 from the center of the Waverly road. Thence, N 85° 15′ W to the center of the Scranton turnpike. The remaining boundaries of the Lenox lot are formed by the center lines of the adjoining highways. By prolonging the boundary between Henry Watson and James Lenox northward 275 feet, we form the east boundary of John Andrews's lot. The same line prolonged to the center of Hall road forms the east boundary of Clayton Andrews's lot. The boundary between lots of John and Clayton Andrews is formed by the center line of the lane, and that center line produced to the east boundary of the lots. The remaining boundaries of these lots are formed by the center lines of the adjacent highways. At 210 feet from intersection *C*, as measured on the center line of the Waverly road and Lenox lane, is the northwest corner of a lot belonging to the Lenox estate. This lot has a front of 60 feet and a depth of 180 feet, the sides being at right angles to the center line of Lenox lane.

All buildings the student will locate by eye, giving to them the same shape and proportions as shown in the plate. Shade trees are spaced 50 feet, their rows being placed 10 feet inside the road boundaries. Fruit trees are spaced 40 and 30 feet. The usual conventional signs are employed to represent the topography. As grass and cereals are much alike in appearance, the conventional sign for grass may be varied so as to represent them all and so give variety to the drawing. A part of the lot belonging to James Henderson is occupied by a vineyard, which is represented by rectangles enclosed in wavy outlines. These signs might also be used to represent small fruits growing on trellises. All other conventional signs employed have been previously described, with appropriate illustrations.

1386. Colored Topography.—All conventional signs so far described are made with a pen. Often, where surveys cover extensive areas, the labor and time for pen work can not be spared, and colors applied with a brush are used instead. With a skilful hand, work of this character may be rapid and very effective. But three colors besides India ink are required; gamboge (yellow), indigo (blue), and lake (scarlet). The colors used in the drafting room are of two kinds, viz., *dry* and *moist*. Dry colors are sold in the form of rectangular cakes, wrapped in tin foil. Moist colors are packed in small dishes of porcelain. These dishes are rectangular in form, open at top. The surface of the paint is covered with wax and the entire dish wrapped in tinfoil. In using, rub the cake of color with a moistened brush which will take up sufficient color. Dilute the color in water to the proper tint, which should always be light and delicate. To cover a surface with a uniform tint, use a camel's hair or sable brush. Use a separate brush for each tint and provide plenty of dishes for the various colors. Confusion in the use of brushes is sure to spoil a tint. For large masses of the same tint, a large brush should be used, but for vegetation or small details, small brushes are indispensable. Avoid heavy strokes. Light and rapid strokes produce smooth and pleasing effects. The map should be pinned to a light drawing board, so that it may readily be inclined at an angle. Keep the brush well filled with color and begin at the top of the surface, inclining the board towards you. If the outline is very irregular, moisten the edge with water. Apply the tint the full length of the surface and continue it down the surface, *never allowing the edge to dry*, which is the secret of a smooth tint.

Woods are commonly colored yellow; grass land, green, made of gamboge and indigo; cultivated land, brown, made of lake, gamboge, and India ink; brushwood, marbled green and yellow; vineyards, purple, made of lake and indigo; lakes and rivers, of light blue, with a darker tint at the shore line; seas, dark blue, with a little yellow added; marshes, the water blue, with patches of green applied horizontally, and roads, dark brown. Woods may be made very effective by drawing the trees, coloring the angle towards the light (the upper left hand) with a touch of yellow, and indigo on the opposite, or lower, right-hand side.

Skill and judgment in mixing and applying colors can be acquired only by practice. When a combination tint, such as brown, is required, the draftsman must estimate how much coloring is required and provide accordingly. He is liable to use too much color producing a heavy tint, which is almost certain to become streaked when applied. A separate brush should be used to take up each color, the brush being moistened and rubbed on the cake. A tinting dish of either glass or porcelain contains the water. The brushes carrying the colors are dipped into the water, each giving off its proportion of color. The water is then stirred until every particle of color is dissolved. If the tint is too light, add more color; if too heavy—a common fault—add more water until the proper shade is obtained. Any tint is deepened by repeating the application of it. When a tint is to be shaded from light to dark, give the entire surface one coat, which will give the lightest shade. Decide how many coats are to be applied to produce the deepest tint and divide the length of the surface into the same number of equal spaces. Beginning at the top of the surface to be shaded, apply a second coat, stopping one space from the bottom. Then take a clean brush and dip it into clear water and wash the edge of the second coat at its finishing line, brushing downwards, taking up in the brush all excess of coloring matter. Another coat is then applied, commencing again at the top and stopping two spaces from the bottom, washing down the edge with clear water. The paper must not be allowed to dry between the successive applications of the tint. If from any cause it should become dry, the entire surface must be moistened with clear water before another application of the tint. Careful practice will enable the student to produce a smooth tint. When a marbled effect is desired, first cover the entire surface with one tint and then apply the other in shorter or longer strokes of the brush, according to the effect which it is desired to produce.

In tinting shore lines, first trace the outline of the shore with a brush moistened with clear water, extending the wash as far as the tint is to be used. Prepare a color dark blue in shade. Next dip a fine brush in the color and trace the outline of the shore. The adjoining paper being moist will cause the color to run. Then moisten a brush in clear water and wash the shore line, the strokes of the brush being drawn from the shore. The effect will be a dark blue shore line shaded to light blue. The dark brown color for roads is produced by adding India ink to the brown representing grass land.

1387. Scales. — The scale of a topographical map should depend upon the character of the work involved, but should always be large enough to clearly admit all necessary details without making the map unwieldy. The work should be so well done that dimensions may be accurately scaled from the map without any calculation. For small plats, such as public squares and small parks, 50 feet to the inch would be a suitable scale, admitting the smallest detail. For larger areas, such as town sites, extensive parks, suburban resorts, etc., a scale of 200 feet to the inch is commonly used. The scale must be reduced in proportion as the area is increased. Published topographical maps are usually made to a scale of one inch to the mile, admitting of the representation of all towns, villages, farms, woods, isolated buildings, and every stream of 600 feet in length, and every hill of 100 feet in height and 500 or 600 feet in horizontal extent.

On a scale of two inches to the mile, the various features of the ground can be clearly and accurately represented. All streams of 300 feet in length, every pond not less than 50 feet broad, and all towns, villages, roads, foot-paths, farms, and isolated buildings.

A scale of six inches to the mile is best for military purposes, admitting of a complete delineation of a country. In all cases the character of the surface and the purpose of the map should determine the scale.

1388. Size of Maps.—Maps for use in the field may vary in size from 18 by 24 inches to 24 by 30 inches. Both sizes are suitable for railroad work. The lines should be so arranged on the different sheets that they may be fitted together, making a continuous map of the line of survey. The sheets should be numbered in regular rotation, and when pinned together they will appear as shown in Fig. 354.

Where possible, arrange the sheets so that each curve with its center and limiting radii may come on the same sheet. Sometimes this can not be done. The points where the different sheets join on to each other should be fixed by a line drawn at right angles to the center line or radial line

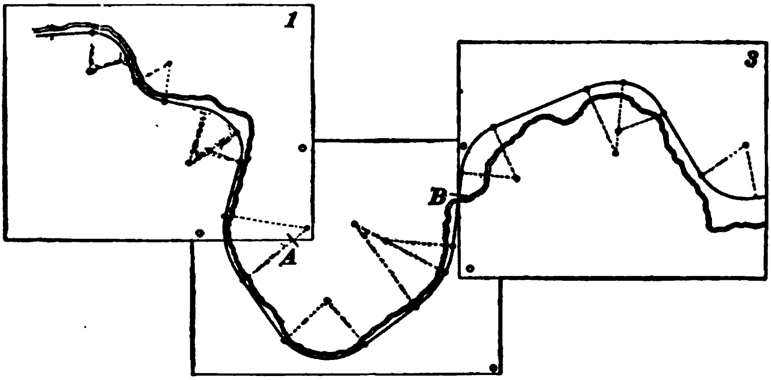

FIG. 354.

at the point of junction, as at *A* or *B*. This simplifies the work of fitting the sheets and greatly promotes accuracy.

1389. Lettering.—Legibility and uniformity are the requisites for good lettering. Ornamental letters, excepting for titles, are entirely out of place, and they are only admissible for titles of very elaborate maps. All lettering in the body of the map or details should be in italics. Small letters should be two-thirds the height of capitals, ordinary capitals $\frac{1}{8}$ of an inch in height, and small letters $\frac{2}{3}$ of $\frac{1}{8}$ or $\frac{1}{12}$ inch in height. Uniformity in spacing letters is as important as uniformity in size. There is no work where practice is more essential, if skill is to be acquired, and nothing adds more to the finish of a drawing than good lettering, while poor and slovenly lettering will rob of all merit an otherwise perfect drawing.

1390. General Instructions.—If the entire map is to be contained on a single sheet, judgment is required in fixing the direction of the first course so as to attain that result. The points of the compass must also be in their natural order, viz., *North* at the top of the map and *South* at the bottom.

The outline of the map will determine the position of the title. Very *fine* lines are a blemish rather than a merit, and heavy lines are likewise to be avoided except when used for shading or boundaries. Boundaries of private property are represented by bold, full lines, and those of state, county, or municipality by heavy broken and dotted lines. All dimensions should be expressed in figures, and all important lines and objects briefly but accurately described.

A SERIES

OF

QUESTIONS AND EXAMPLES

RELATING TO THE SUBJECTS TREATED OF IN THIS VOLUME.

It will be noticed that the questions and examples contained in the following pages are divided into sections corresponding to the sections of the text of the preceding pages, and that each section has a headline which is the same as the headline of the section to which the questions refer. No attempt should be made to answer any questions or to work any examples until the corresponding part of the text has been carefully studied.

ARITHMETIC.

(SECTION 1.)

EXAMINATION QUESTIONS.

(1) What is arithmetic?

(2) What is a number?

(3) What is the difference between a concrete number and an abstract number?

(4) Define notation and numeration.

(5) Write each of the following numbers in words:

(*a*) 980; (*b*) 605; (*c*) 28,284; (*d*) 9,006,042; (*e*) 850,317,002; (*f*) 700,004.

(6) Represent in figures the following expressions:

(*a*) Seven thousand, six hundred. (*b*) Eighty-one thousand, four hundred, two. (*c*) Five million, four thousand, seven. (*d*) One hundred eight million, ten thousand, one. (*e*) Eighteen million, six. (*f*) Thirty thousand, ten.

(7) What is the sum of $3,290 + 504 + 865,403 + 2,074 + 81 + 7$? Ans. 871,359.

(8) $709 + 8,304,725 + 391 + 100,302 + 300 + 909 =$ what? Ans. 8,407,336.

(9) Find the difference between the following:

(*a*) 50,962 and 3,338; (*b*) 10,001 and 15,339.

Ans. $\begin{cases} (a) & 47,624. \\ (b) & 5,338. \end{cases}$

(10) (*a*) $70,968 - 32,975 =$? (*b*) $100,000 - 98,735 =$?

Ans. $\begin{cases} (a) & 37,993. \\ (b) & 1,265. \end{cases}$

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(11) The greater of two numbers is 1,004 and their difference is 49; what is their sum? Ans. 1,959.

(12) From 5,962 + 8,471 + 9,023 take 3,874 + 2,039.
Ans. 17,543.

(13) A man willed \$125,000 to his wife and two children; to his son he gave \$44,675, to his daughter \$26,380, and to his wife the remainder. What was his wife's share?
Ans. \$53,945.

(14) Find the products of the following:
(*a*) 526,387 × 7; (*b*) 700,298 × 17; (*c*) 217 × 103 × 67.

Ans. (*a*) 3,684,709.
(*b*) 11,905,066.
(*c*) 1,497,517.

(15) If your watch ticks once every second, how many times will it tick in one week? Ans. 604,800 times.

(16) If a monthly publication contains 24 pages in each issue, how many pages will there be in eight yearly volumes?
Ans. 2,304.

(17) An engine and boiler in a manufactory are worth \$3,246. The building is worth three times as much, plus \$1,200, and the tools are worth twice as much as the building, plus \$1,875. (*a*) What is the value of the building and tools? (*b*) What is the value of the whole plant?

Ans. (*a*) \$34,689.
(*b*) \$37,935.

(18) Solve the following by cancelation:

$$(a)\ \frac{72 \times 48 \times 28 \times 5}{96 \times 15 \times 7 \times 6} = ? \qquad (b)\ \frac{80 \times 60 \times 50 \times 16 \times 14}{70 \times 50 \times 24 \times 20} = ?$$

Ans. (*a*) 8.
(*b*) 32.

(19) If a mechanic earns \$1,500 a year for his labor, and his expenses are \$968 per year, in what time can he save enough to buy 28 acres of land, at \$133 an acre?
Ans. 7 years.

(20) A freight train ran 365 miles in one week, and 3 times as far, lacking 246 miles, the next week; how far did it run the second week? Ans. 849 miles.

(21) If the driving wheel of a locomotive is 16 ft. in circumference, how many revolutions will it make in going from Philadelphia to Pittsburg, the distance of which is 354 miles, there being 5,280 feet in one mile? Ans. 116,820 rev.

(22) What is the quotient of
(*a*) 589,824 ÷ 576? (*b*) 369,730,620 ÷ 43,911? (*c*) 2,527,525 ÷ 505? (*d*) 4,961,794,302 ÷ 1,234?

Ans. (*a*) 1,024.
(*b*) 8,420.
(*c*) 5,005.
(*d*) 4,020,903.

(23) A man paid $444 for a horse, wagon, and harness. If the horse cost $264 and the wagon $153, how much did the harness cost? Ans. $27.

(24) What is the product of
(*a*) 1,024 × 576? (*b*) 5,005 × 505? (*c*) 43,911 × 8,420?

Ans. (*a*) 589,824.
(*b*) 2,527,525.
(*c*) 369,730,620.

(25) If a man receives 30 cents an hour for his wages, how much will he earn in a year, working 10 hours a day and averaging 25 days per month? Ans. $900.

ARITHMETIC.

(SECTION 2.)

EXAMINATION QUESTIONS.

(26) What is a fraction?

(27) What are the terms of a fraction?

(28) What does the denominator show?

(29) What does the numerator show?

(30) How do you find the value of a fraction?

(31) Is $\frac{13}{8}$ a proper or an improper fraction, and why?

(32) Write three mixed numbers.

(33) Reduce the following fractions to their lowest terms: $\frac{4}{8}$, $\frac{4}{16}$, $\frac{8}{32}$, $\frac{32}{64}$. Ans. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{2}$.

(34) Reduce 6 to an improper fraction whose denominator is 4. Ans. $\frac{24}{4}$.

(35) Reduce $7\frac{7}{8}$, $13\frac{5}{16}$, and $10\frac{3}{4}$ to improper fractions. Ans. $\frac{63}{8}$, $\frac{213}{16}$, $\frac{43}{4}$.

(36) What is the value of each of the following: $\frac{13}{2}$, $\frac{17}{4}$. $\frac{69}{16}$. $\frac{16}{8}$, $\frac{67}{64}$? Ans. $6\frac{1}{2}$, $4\frac{1}{4}$, $4\frac{5}{16}$, 2, $1\frac{3}{64}$.

(37) Solve the following:

(*a*) $35 \div \frac{5}{16}$; (*b*) $\frac{9}{16} \div 3$; (*c*) $\frac{17}{2} \div 9$; (*d*) $\frac{113}{64} \div \frac{7}{16}$; (*e*) $15\frac{3}{4} \div 4\frac{3}{8}$.

Ans.
- (*a*) 112.
- (*b*) $\frac{3}{16}$.
- (*c*) $\frac{17}{18}$.
- (*d*) $4\frac{1}{28}$.
- (*e*) $3\frac{3}{5}$.

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(38) $\frac{1}{8}+\frac{3}{8}+\frac{4}{8}=$? Ans. 1.

(39) $\frac{1}{4}+\frac{3}{8}+\frac{5}{16}=$? Ans. $\frac{15}{16}$.

(40) $42+31\frac{5}{8}+9\frac{7}{16}=$? Ans. $83\frac{1}{16}$.

(41) An iron plate is divided into four sections; the first contains $29\frac{3}{4}$ square inches; the second, $50\frac{5}{8}$ square inches; the third, 41 square inches; and the fourth, $69\frac{3}{16}$ square inches. How many square inches are in the plate?

Ans. $190\frac{9}{16}$ sq. in.

(42) Find the value of each of the following:

(*a*) $\frac{7}{\frac{3}{16}}$; (*b*) $\frac{\frac{15}{32}}{\frac{5}{8}}$; (*c*) $\frac{\frac{4+3}{2+6}}{5}$.

Ans. $\begin{cases} (a)\ 37\frac{1}{3}. \\ (b)\ \frac{3}{4}. \\ (c)\ \frac{7}{40}. \end{cases}$

(43) The numerator of a fraction is 28, and the value of the fraction $\frac{7}{8}$; what is the denominator? Ans. 32.

(44) What is the difference between (*a*) $\frac{7}{8}$ and $\frac{7}{16}$? (*b*) 13 and $7\frac{7}{16}$? (*c*) $312\frac{9}{16}$ and $229\frac{5}{32}$?

Ans. $\begin{cases} (a)\ \frac{7}{16}. \\ (b)\ 5\frac{9}{16}. \\ (c)\ 83\frac{13}{32}. \end{cases}$

(45) If a man travels $85\frac{5}{12}$ miles in one day, $78\frac{9}{16}$ miles in another day, and $125\frac{11}{36}$ miles in another day, how far did he travel in the three days? Ans. $289\frac{41}{144}$ miles.

(46) From $573\frac{4}{5}$ tons take $216\frac{5}{8}$ tons. Ans. $357\frac{7}{40}$.

(47) At $\frac{3}{8}$ of a dollar a yard, what will be the cost of $9\frac{1}{4}$ yards of cloth? Ans. $3\frac{15}{32}$ dollars.

(48) Multiply $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{7}{11}$ of $\frac{19}{20}$ of 11 by $\frac{7}{8}$ of $\frac{5}{6}$ of 45.

Ans. $109\frac{13}{128}$.

(49) How many times are $\frac{2}{3}$ contained in $\frac{3}{4}$ of 16?

Ans. 18 times.

(50) Bought $211\frac{1}{4}$ pounds of old lead for $1\frac{7}{8}$ cents per pound. Sold a part of it for $2\frac{1}{2}$ cents per pound, receiving for it the same amount as I paid for the whole. How many pounds did I have left? Ans. $52\frac{13}{16}$ pounds.

ARITHMETIC.

(SECTION 3.)

EXAMINATION QUESTIONS.

(51) Write out in words the following numbers: .08, .131, .0001, .000027, .0108, and 93.0101.

(52) How do you place decimals for addition and subtraction ?

(53) Give a rule for multiplication of decimals.

(54) Give a rule for division of decimals.

(55) State the difference between a fraction and a decimal.

(56) State how to reduce a fraction to a decimal.

(57) Reduce the following fractions to equivalent decimals: $\frac{1}{2}$, $\frac{7}{8}$, $\frac{5}{32}$, $\frac{65}{100}$, and $\frac{125}{1000}$.

$$\text{Ans.}\begin{cases} .5. \\ .875. \\ .15625. \\ .65 \\ .125 \end{cases}$$

(58) Solve the following: (*a*) $\frac{32.5 + .29 + 1.5}{4.7 + 9}$;

(*b*) $\frac{1.283 \times \overline{8 + 5}}{2.63}$; (*c*) $\frac{\overline{589 + 27} \times \overline{163 - 8}}{25 + 39}$;

(*d*) $\frac{\overline{40.6 + 7.1} \times (3.029 - 1.874)}{6.27 + 8.53 - 8.01}$.

$$\text{Ans.}\begin{cases} (a)\ 2.5029. \\ (b)\ 6.3418. \\ (c)\ 1{,}491.875 \\ (d)\ 8.1139. \end{cases}$$

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(59) How many inches in .875 of a foot? Ans. $10\frac{1}{2}$ in.

(60) What decimal part of a foot is $\frac{3}{16}$ of an inch?
Ans. .015625.

(61) A cubic inch of water weighs .03617 of a pound. What is the weight of a body of water whose volume is 1,500 cubic inches? Ans. 54.255 lb.

(62) If by selling a carload of coal for $82.50, at a profit of $1.65 per ton, I make enough to pay for 72.6 ft. of fencing at $.50 a foot, how many tons of coal were in the car?
Ans. 22 tons.

(63) Divide 17,892 by 231, and carry the result to four decimal places. Ans. 77.4545+.

(64) Find the value of the following expression when the result is carried to three decimal places:

$$\frac{74.26 \times 24 \times 3.1416 \times 19 \times 19 \times 350}{33,000 \times 12 \times 4} = ? \quad \text{Ans. } 446.619-.$$

(65) Express (*a*) .7928 in 64ths; (*b*) .1416 in 32ds; (*c*) .47915 in 16ths.

Ans.
(*a*) $\frac{51}{64}$.
(*b*) $\frac{5}{32}$.
(*c*) $\frac{8}{16}$.

(66) Work out the following examples:

(*a*) 709.63 − .8514; (*b*) 81.963 − 1.7; (*c*) 18 − .18; (*d*) 1 − .001; (*e*) 872.1 − (.8721 + .008); (*f*) (5.028 + .0073) − (6.704 − 2.38).

Ans.
(*a*) 708.7786.
(*b*) 80.263.
(*c*) 17.82.
(*d*) .999.
(*e*) 871.2199.
(*f*) .7113.

(67) Work out the following:

(*a*) $\frac{7}{8}$ − .807; (*b*) .875 − $\frac{3}{8}$; (*c*) ($\frac{5}{32}$ + .435) − ($\frac{21}{100}$ − .07); (*d*) what is the difference between the sum of 33-millionths and 17-thousandths, and the sum of 53-hundredths and 274-thousandths?

Ans. (a) .068.
(b) .5.
(c) .45125.
(d) .786967.

(68) What is the sum of .125, .7, .089, .4005, .9, and .000027? Ans. 2.214527.

(69) $927.416 + 8.274 + 372.6 + 62.07938 = ?$
Ans. 1,370.36938.

(70) Add 17-thousandths, 2-tenths, and 47-millionths.
Ans. .217047.

(71) Find the products of the following expressions:

(a) $.013 \times .107$; (b) $203 \times 2.03 \times .203$; (c) $2.7 \times 31.85 \times (3.16 - .316)$; (d) $(107.8 + 6.541 - 31.96) \times 1.742$.

Ans. (a) .001391.
(b) 83.65427.
(c) 244.56978.
(d) 143.507702.

(72) Solve the following:

(a) $(\frac{7}{16} - .13) \times \overline{.625 + \frac{5}{8}}$; (b) $(\frac{19}{32} \times .21) - (.02 \times \frac{3}{16})$; (c) $(\frac{13}{4} + .013 - 2.17) \times \overline{13\frac{1}{4} - 7\frac{5}{16}}$.

Ans. (a) .384375.
(b) .1209375.
(c) 6.4896875.

(73) Solve the following:

(a) $.875 \div \frac{1}{2}$; (b) $\frac{7}{8} \div .5$; (c) $\dfrac{.375 \times \frac{1}{4}}{\frac{5}{16} - .125}$.

Ans. (a) 1.75.
(b) 1.75.
(c) .5.

(74) Find the value of the following expression:

$$\frac{\dfrac{1.25 \times 20 \times 3}{87 + (11 \times 8)}}{459 + 32}.$$

Ans. $210\frac{3}{7}$.

(75) From 1 plus .001 take .01 plus .000001.
Ans. .990999

ARITHMETIC.

(SECTION 4.)

EXAMINATION QUESTIONS.

(76) What is 25 per cent. of 8,428 lb.? Ans. 2,107 lb.

(77) What is 1 per cent. of $100? Ans. $1.

(78) What is ½ per cent. of $35,000? Ans. $175.

(79) What per cent. of 50 is 2? Ans. 4%.

(80) What per cent. of 10 is 10? Ans. 100%.

(81) Solve the following:

(*a*) Base = $2,522 and percentage = $176.54. What is the rate? (*b*) Percentage = 16.96 and rate = 8 per cent. What is the base? (*c*) Amount = 216.7025 and base = 213.5. What is the rate? (*d*) Difference = 201.825 and base = 207. What is the rate?

Ans. (*a*) 7%.
(*b*) 212.
(*c*) 1½%.
(*d*) 2½%.

(82) A farmer gained 15% on his farm by selling it for $5,500. What did it cost him? Ans. $4,782.61.

(83) A man receives a salary of $950. He pays 24% of it for board, 12½% of it for clothing, and 17% of it for other expenses. How much does he save in a year? Ans. $441.75.

(84) If 37½ per cent. of a number is 961.38, what is the number? Ans. 2,563.68.

(85) A man owns ¾ of a property. 30% of his share is worth $1,125. What is the whole property worth?

Ans. $5,000

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(86) What sum diminished by 35% of itself equals $4,810? Ans. $7,400.

(87) A merchant's sales amounted to $197.55 on Monday, and this sum was 12½% of his sales for the week. How much were his sales for the week? Ans. $1,580.40.

(88) The distance between two stations on a certain railroad is 16.5 miles, which is 12½% of the entire length of the road. What is the length of the road? Ans. 132 miles.

(89) After paying 60% of my debts, I find that I still owe $35. What was my whole indebtedness? Ans. $87.50.

(90) Reduce 28 rd. 4 yd. 2 ft. 10 in. to inches. Ans. 5,722 in.

(91) Reduce 5,722 in. to higher denominations. Ans. 28 rd. 4 yd. 2 ft. 10 in.

(92) How many seconds in 5 weeks and 3.5 days? Ans. 3,326,400 sec.

(93) How many pounds, ounces, pennyweights, and grains are contained in 13,750 gr.? Ans. 2 lb. 4 oz. 12 pwt. 22 gr.

(94) Reduce 4,763,254 links to miles. Ans. 595 mi. 32 ch. 54 li.

(95) Reduce 764,325 cu. in. to cu. yd. Ans. 16 cu. yd. 10 cu. ft. 549 cu. in.

(96) What is the sum of 2 rd. 2 yd. 2 ft. 3 in.; 4 yd. 1 ft. 9 in.; 2 ft. 7 in.? Ans. 3 rd. 2 yd. 2 ft. 1 in.

(97) What is the sum of 3 gal. 3 qt. 1 pt. 3 gi.; 6 gal. 1 pt. 2 gi.; 4 gal. 1 gi.; 8 qt. 5 pt.? Ans. 16 gal. 3 qt. 2 gi.

(98) What is the sum of 240 gr. 125 pwt. 50 oz. and 3 lb.? Ans. 7 lb. 8 oz. 15 pwt.

(99) What is the sum of 11° 16′ 12″; 13° 19′ 30″; 20° 25″; 26′ 29″; 10° 17′ 11″? Ans. 55° 19′ 47″.

(100) What is the sum of 130 rd. 5 yd. 1 ft. 6 in.; 215 rd. 2 ft. 8 in.; 304 rd. 4 yd. 11 in.? Ans. 2 mi. 10 rd. 5 yd. 7 in.

(101) What is the sum of 21 A. 67 sq. ch. 3 sq. rd. 21 sq. li.; 28 A. 78 sq. ch. 2 sq. rd. 23 sq. li.; 47 A. 6 sq. ch. 2 sq.

rd. 18 sq. li.; 56 A 59 sq. ch. 2 sq. rd. 16 sq. li.; 25 A. 38 sq. ch. 3 sq. rd. 23 sq. li.; 46 A. 75 sq. ch. 2 sq. rd. 21 sq. li.?
Ans. 255 A. 3 sq. ch. 14 sq. rd. 122 sq. li.

(102) From 20 rd. 2 yd. 2 ft. 9 in. take 300 ft.
Ans. 2 rd. 1 yd. 2 ft. 9 in.

(103) From a farm containing 114 A. 80 sq. rd. 25 sq. yd., 75 A. 70 sq. rd. 30 sq. yd. are sold. How much remains?
Ans. 39 A. 9 sq. rd. 25¼ sq. yd.

(104) From a hogshead of molasses, 10 gal. 2 qt. 1 pt. are sold at one time, and 26 gal. 3 qt. at another time. How much remains? Ans. 25 gal. 2 qt. 1 pt.

(105) If a person were born June 19, 1850, how old would he be August 3, 1892? Ans. 42 yr. 1 mo. 14 da.

(106) A note was given August 5, 1890, and was paid June 3, 1892. What length of time did it run?
Ans. 1 yr. 9 mo. 28 da.

(107) What length of time elapsed from 16 min. past 10 o'clock A. M., July 4, 1883, to 22 min. before 8 o'clock P. M., Dec. 12, 1888? Ans. 5 yr. 5 mo. 8 da. 9 hr. 22 min.

(108) If 1 iron rail is 17 ft. 3 in. long, how long would 51 rails be, if placed end to end? Ans. 53 rd. 1½ yd. 9 in.

(109) Multiply 3 qt. 1 pt. 3 gi. by 4.7.
Ans. 4 gal. 2 qt. 1.7 gi.

(110) Multiply 3 lb. 10 oz. 13 pwt. 12 gr. by 1.5.
Ans. 5 lb. 10 oz. 6 gr.

(111) How many bushels of apples are contained in bbl., if each barrel contains 2 bu. 3 pk. 6 qt.?
Ans. 26 bu. 1 pk. 6 qt.

(112) Multiply 7 T. 15 cwt. 10.5 lb. by 1.7.
Ans. 13 T. 3 cwt. 67.85 lb.

(113) Divide 358 A. 57 sq. rd. 6 sq. yd. 2 sq. ft. by 7.
Ans. 51 A. 31 sq. rd. 8 sq. ft.

(114) Divide 282 bu. 3 pk. 1 qt. 1 pt. by 12.
Ans. 23 bu. 2 pk. 2 qt. ¼ pt.

(115) How many iron rails, each 30 ft. long, are required to lay a railroad track 23 miles long? Ans. 8,096 rails.

(116) How many boxes, each holding 1 bu. 1 pk. and 7 qt., can be filled from 356 bu. 3 pk. and 5 qt. of cranberries? Ans. 243 boxes.

(117) If 16 square miles are equally divided into 62 farms, how much land will each contain?

Ans. 165 A. 25 sq. rd. 24 sq. yd. 3 sq. ft. 80 + sq. in.

ARITHMETIC.

(SECTION 5.)

EXAMINATION QUESTIONS.

(118) What is the square of 108? Ans. 11,664.

(119) Find the fifth power of 9. Ans. 59,049.

(120) What is the value of $.0133^3$? Ans. .000002352637.

(121) In what respect does evolution differ from involution?

(122) Extract the square root of 90. Ans. 9.4868+.

(123) Find the value of $(3\frac{3}{4})^3$. Ans. $52\frac{47}{64}$, or 52.734375.

(124) What is the cube root of 92,416? Ans. 45.212−.

(125) Find the value of $\sqrt{502,681}$. Ans. 709.

(126) What is the value of $\sqrt[3]{\frac{27}{64}}$? Ans. $\frac{3}{4}$.

(127) From the cube of 4 take the cube root of 8. Ans. 62.

(128) What is the value of $\sqrt[3]{\frac{3}{8}}$? Ans. .72112+.

(129) Extract the square root of .3364. Ans. .58.

(130) Find the square root of 3.1416. Ans. 1.7725−.

(131) What number multiplied by itself equals 114.9184? Ans. 10.72.

(132) Extract the square root of 3,486,784. Ans. 1,867.3−.

(133) Find the square root of .00041209. Ans. .0203.

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ARITHMETIC.

(SECTION 6.)

EXAMINATION QUESTIONS.

Find the value of x in the following:

(143) $11.7 : 13 :: 20 : x$. Ans. 22.22+.

(144) (*a*) $20 + 7 : 10 + 8 :: 3 : x$; (*b*) $12^2 : 100^2 :: 4 : x$. Ans. $\begin{cases} (a)\ 2. \\ (b)\ 277.7+. \end{cases}$

(145) (*a*) $\frac{4}{x} = \frac{7}{21}$; (*b*) $\frac{x}{24} = \frac{8}{16}$; (*c*) $\frac{2}{10} = \frac{x}{100}$; (*d*) $\frac{15}{45} = \frac{60}{x}$; (*e*) $\frac{10}{150} = \frac{x}{600}$. Ans. (*a*) $x = 12$; (*b*) $x = 12$; (*c*) $x = 20$; (*d*) $x = 180$; (*e*) $x = 40$.

(146) $x : 5 :: 27 : 12.5$. Ans. $10\frac{4}{5}$.

(147) $45 : 60 :: x : 24$. Ans. 18.

(148) $x : 35 :: 4 : 7$. Ans. 20.

(149) $9 : x :: 6 : 24$. Ans. 36.

(150) $\sqrt[3]{1,000} : \sqrt[3]{1,331} = 27 : x$. Ans. 29.7.

(151) $64 : 81 = 21^2 : x^2$. Ans. 23.625.

(152) $7 + 8 : 7 = 30 : x$. Ans. 14.

(153) A man whose steps measure 2 ft. 5 in. takes 2,480 steps in walking a certain distance. How many steps of 2 ft. 7 in. will be required for the same distance? Ans. 2,320 steps.

(154) If a horse travels 12 mi. in 1 hr. 36 min., how far will he travel at the same rate in 15 hr.? Ans. 112.5 mi.

(155) If a column of mercury 27.63 in. high weighs .76 of a pound, what will be the weight of a column of mercury having the same diameter, 29.4 inches high? Ans. .808+lb.

(156) If 2 gal. 3 qt. 1 pt. of water will last a man 5 da., how long will 5 gal. 3 qt. last him, if he drinks at the same rate? Ans. 10 da.

(157) Heat from a burning body varies inversely as the square of the distance from it. If a thermometer held 6 ft.

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from a stove shows a rise in temperature of 24 degrees, how many degrees rise in temperature would it indicate if held 12 ft. from the stove? Ans. 6°.

(158) If a pile of wood 12 ft. long, 4 ft. wide, and 3 ft. high is worth $12, what is the value of a pile of wood 15 ft. long, 5 ft. wide, and 6 ft. high? Ans. $37.50.

(159) If 100 gal. of water run over a dam in 2 hr., how many gallons will run over the dam in 14 hr. 28 min.? Ans. $723\frac{1}{3}$ gal.

(160) If a cistern 28 ft. long, 12 ft. wide, 10 ft. deep holds 510 bbl. of water, how many barrels of water will a cistern hold that is 20 ft. long, 17 ft. wide, and 6 ft. deep? Ans. $309\frac{9}{14}$ bbl.

(161) If a railway train runs 444 mi. in 8 hr. 40 min., in what time can it run 1,060 mi. at the same rate of speed? Ans. 20 hr. 41.44 min.

(162) If sound travels at the rate of 6,160 ft. in $5\frac{1}{2}$ sec., how far does it travel in 1 min.? Ans. 67,200 ft.

(163) If 5 men by working 8 hours a day can do a certain amount of work, how many men by working 10 hours a day can do the same work? Ans. 4 men.

(164) If a man travel 540 miles in 20 days of 10 hours each, how many hours a day must he travel to cover 630 miles in 25 days? Ans. $9\frac{1}{3}$ hr.

(165) Referring to example 4, Art. **349**, what is the horsepower of an engine whose cylinder is 30 in. in diameter, piston speed 660 ft. per min., and mean effective pressure 42 lb. per sq. in.? Ans. 594 horsepower.

(166) The weight of a cubic inch of cast iron is .261 pound. Referring to Art. **345**, what is the weight of a solid cast-iron cylinder whose diameter is 12 inches and length is 60 inches? Ans. 1,771.11 lb.

(167) Referring to Art. **348**, what is the centrifugal force of a 40-pound body revolving in a circle having a radius of 10 inches, at a speed of 18 feet per second? Ans. 484.7 lb

FORMULAS.

(ARTS. 350–355.)

EXAMINATION QUESTIONS.

$A = 5$ $\quad h = 200$
$B = 10$ $\quad x = 12$
$i = 3.5$ $\quad D = 120$

Work out the solutions to the following formulas, using the above values for the letters:

(168) $C = \frac{D - x}{B + i}$. Ans. $C = 8$.

(169) $Q = \frac{A\,h + D}{2\,x + 6} + D$. Ans. $Q = 157\frac{1}{3}$.

(170) $r = \frac{3.246\,B\,h}{\frac{A\,x + h}{A\,i - b}}$. Ans. $r = 187.269+$.

(171) $v = \sqrt{\frac{A\,D}{i\,B + 1.5}}$. Ans. $v = 4.05+$.

(172) $u = \sqrt[3]{\frac{B\,x}{.00018\,h\,(A^2 - x)}}$. Ans. $u = 6.35+$.

(173) $f = \frac{10\,(h - D)^2}{\sqrt[3]{D + A}}$. Ans. $f = 12{,}800$.

(174) $g = \frac{(B - A)^2 - \sqrt[3]{D + A}}{A^3 - (1 + D)}$. Ans. $g = 5$.

(175) $k = \sqrt{\frac{A\,B^2}{\sqrt[3]{A\,h}}}$. Ans. $k = 7.071+$.

(176) $T = \sqrt{\frac{A^2\left[490 + \frac{(h\,x)^2}{D^2}\right]}{h + \frac{x}{D}(A^2 - B)^2}}$. Ans. $T = 10$.

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GEOMETRY AND TRIGONOMETRY

EXAMINATION QUESTIONS

NOTE.—In solving the following examples, the student will find that he will understand them much better if he draws a diagram for each, showing the given conditions and results sought.

(1) If one of the angles formed by one straight line meeting another straight line equals $\frac{5}{6}$ of a right angle, what is the other angle equal to? Ans. $1\frac{1}{6}$ right angles.

(2) If a triangle has two equal angles, what kind of a triangle is it?

(3) The perimeter of a regular decagon is 40 inches; what is the length of a side? Ans. 4 in.

(4) What is one angle of a regular dodecagon equal to? Ans. $1\frac{2}{3}$ right angles.

(5) A triangle has three equal angles; what is it called?

(6) A certain triangle has two equal angles. If, from the vertex of the other angle, a perpendicular is drawn to the side opposite, which is 7 inches long, what are the lengths of the two parts of the side thus divided by the perpendicular?

(7) The shortest distance from a given point to a given line is 9 inches; the distances from this point to the two extremities of the line are 12 inches and 15 inches; what is the length of the line? Ans. 19.94 in.

(8) What is one of the angles of an equiangular octagon equal to? Ans. $1\frac{1}{2}$ right angles.

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(9) Given three points A, B, and C, and the distance from A to B equal to $1\frac{1}{2}$ inches, from B to C $1\frac{1}{2}$ inches, and from C to A 2 inches; pass a circle through these three points.

(10) The chord of an arc in a circle whose radius is 6 inches is 4 inches long; what is the length of the chord of half the arc? Ans. 2.03 in.

(11) The length of a perpendicular from the center of a circle to a chord is $5\frac{3}{4}$ inches; if the diameter of the circle is 17 inches, what is the length of the chord? Ans. 12.52 in.

(12) The sides of an inscribed angle intercept three-fourths of the circumference; how many quadrants are there in the angle? Ans. $1\frac{1}{2}$ quadrants.

(13) How many equal sectors are there in a circle, if each sector measures $\frac{2}{7}$ of a right angle? Ans. 14 sectors.

(14) If the perimeter of a regular inscribed octagon is 24 inches and the length of the perpendicular from the center to one of the sides is 3.62 inches, what is the diameter of the circle in which the octagon is inscribed?

Ans. 7.84 in.

(15) What part of a circle is an arc of 19° 19′ and 19″? Express it decimally. Ans. .053672 of a circle.

(16) In a triangle ABC, $AB = 26$ feet 7 inches, $AC = 40$ feet, and the included angle $A = 36° 20' 43''$; find the remaining parts.

$$\text{Ans.}\begin{cases} C = 40° 16' 52''. \\ B = 103° 22' 25''. \\ BC = 24 \text{ ft. } 4.4 \text{ in.} \end{cases}$$

(17) In a triangle ABC, the side $AB = 16$ feet 5 inches, the side $BC = 13$ feet $6\frac{1}{2}$ inches, and the angle $A = 54° 54' 54''$; find the remaining parts.

$$\text{Ans.}\begin{cases} B = 42° 19' 36'', \text{ or} \\ \qquad 27° 50' 36''. \\ C = 82° 45' 30'', \text{ or} \\ \qquad 97° 14' 30''. \\ AC = 11 \text{ ft. } 1\frac{3}{4} \text{ in.,} \\ \qquad \text{or } 7 \text{ ft. } 8\frac{3}{4} \text{ in.} \end{cases}$$

(18) If one-third of an angle of a certain triangle = 14° 47′ 10″, what are the angles, one of the other two being two and one-half times the given angle?

$$\text{Ans.} \begin{cases} 24^\circ\ 44'\ 45''. \\ 44^\circ\ 21'\ 30''. \\ 110^\circ\ 53'\ 45''. \end{cases}$$

(19) In a right triangle $A\,B\,C$, the two sides are 437 feet and 792 feet in length; find the hypotenuse and the two acute angles?

$$\text{Ans.} \begin{cases} 28^\circ\ 53'\ 19''. \\ 61^\circ\ 6'\ 41''. \\ 904 \text{ ft. } 6\tfrac{3}{4} \text{ in.} \end{cases}$$

(20) In a triangle $A\,B\,C$, angle $A = 29^\circ\ 21'$, angle $C = 76^\circ\ 44'\ 18''$, and the side $A\,C =$ 31 feet 10 inches; find the other three parts.

$$\text{Ans.} \begin{cases} B\,C = 16 \text{ ft. } 3 \text{ in.} \\ A\,B = 32 \text{ ft. } 3 \text{ in.} \\ B \quad = 73^\circ\ 54'\ 42''. \end{cases}$$

(21) (*a*) The area of a circle is 89.42 square inches; what is its diameter and circumference? (*b*) What is the length of a side of the largest regular hexagon that could be inscribed in it? Ans. (*b*) 5.335 in.

(22) The distance between two parallel sides of a wrought-iron octagon bar is 2 inches; what is the weight of a bar 10 feet long, a cubic inch of wrought iron weighing .282 pound? Ans. 112 lb. 2 oz.

(23) The outside and inside diameters of a cast-iron spherical shell are 16 inches and 12 inches; what is its weight, a cubic inch of cast iron weighing .261 pound?

Ans. 323.61 lb.

(24) The length of an arc of a circle is $5\frac{13}{32}$ inches by measurement. If the number of degrees in the arc is 27, what is the diameter of the circle? Ans. 22.95 in.

(25) (*a*) What is the area of a circle whose diameter is $17\frac{1}{64}$ inches? (*b*) What is the length of an arc of 16° 7′ 21″ in the above circle? Ans. (*b*) 2.394 in.

(26) (*a*) What is the area of an ellipse whose axes are 12 inches and 8 inches? (*b*) What is its perimeter?

$$\text{Ans.} \begin{cases} (a) \quad 75.4 \text{ sq. in.} \\ (b) \quad 31.731 \text{ in.} \end{cases}$$

(27) What is the entire surface of a cone whose base is 7 inches in diameter and whose altitude is 11 inches?

Ans. 165.41 sq. in.

(28) What is the height of a cone having the same volume and diameter as a 10-inch sphere? Ans. 20 in.

(29) What is the height of a cylinder having the same volume and diameter as a 12-inch sphere? Ans. 8 in.

(30) (*a*) What is the area of a triangle whose base is $9\frac{1}{2}$ inches and whose altitude is 12 inches? (*b*) If the angle which one side forms with the base is 79° 22′, what is the perimeter of the triangle? Ans. (*b*) 35.73 in.

(31) The diagonal of a trapezium is 11 inches; the lengths of the perpendiculars from the opposite vertexes upon this diagonal are $4\frac{1}{2}$ inches and 7 inches; what is the area of the trapezium?

(32) The length of a chord of a segment in a circle whose diameter is 10 inches is $6\frac{3}{4}$ inches; what is the area of the segment and the number of degrees in its arc?

$$\text{Ans.} \begin{cases} 6.074 \text{ sq. in.} \\ 84^\circ\ 54'\ 28.6''. \end{cases}$$

(33) What is the volume and entire area of a frustum of a cone whose upper base is 12 inches and lower base is 18 inches in diameter and whose altitude is 14 inches?

$$\text{Ans.} \begin{cases} 2{,}506.997 \text{ cu. in.} \\ 1{,}042.38 \text{ sq. in.} \end{cases}$$

(34) What is the area of the surface of a sphere 27 inches in diameter? Ans. 2,290.2 sq. in.

(35) What is the volume of an engine cylinder, in cubic feet, whose diameter is 19 inches and whose stroke is 24 inches? Ans. 3.938 cu. ft.

(36) The chord of the arc of a segment is 14 inches long and the height of the segment is 2 inches; what is the radius? Ans. $13\frac{1}{4}$ in.

(37) (*a*) What is the volume of a cylindrical ring whose outside diameter is 16 inches and inside diameter 13 inches? (*b*) If made of cast iron, what is its weight?

Ans. (*b*) 21 lb.

(38) The altitude of a parallelopipedon is 18 inches; its base is a square, one edge measuring 5¼ inches; what is its convex area, entire area, and volume?

Ans. 378 sq. in.
433.125 sq. in.
496.125 cu. in.

(39) What is the convex area and entire area of a hexagonal pyramid, the slant height being 37 feet and one edge of the base measuring 12 feet?

Ans. 1,332 sq. ft.
1,706.112 sq. ft.

(40) If the altitude of the pyramid in the last problem had been 37 feet, what would have been its volume?

Ans. 4,614 cu. ft.

(41) What is the area of a sector if the chord of the arc is 6⅞ inches long and the diameter of the circle is 10 inches?

Ans. 18.95 sq. in.

(42) What is the area in square feet of a parallelogram whose base is 129 inches long, if the shortest distance between the base and side opposite is 7 feet?

(43) The parallel sides of a trapezoid are 15 feet 7 inches and 21 feet 11 inches long; the altitude is 7 feet 8 inches. What is the area of the trapezoid? Ans. 143.75 sq. ft.

(44) What would be (*a*) the length of a side of a square having the same area as the trapezoid in the last problem? (*b*) the diameter of a circle? (*c*) How much shorter is the circumference of the circle than the perimeter of the square?

Ans. (*a*) 11.99 ft.
(*b*) 13½ ft.
(*c*) 5 ft. 6.6 in.

(45) In a triangle ABC, $AB = 24$ feet, $BC = 11$ feet 3 inches, and $AC = 18$ feet; required, the three angles.

Ans. $A = 26° 28' 5''$.
$B = 45° 29' 23''$.
$C = 108° 2' 32''$

SURVEYING.

(ARTS. 1180–1308.)

EXAMINATION QUESTIONS.

Note.—The examples given in this subject are similar to those arising in field and office practice, and the answers given have the same degree of accuracy as required in practical operations. When the answer contains a decimal fraction, the student should carry out the result to one place greater than required in the answer given. If the figure so obtained is less than 5, it is ignored; if it is 5 or greater, add 1 to the preceding figure for the correct answer.

(614) In a triangle whose angles are A, B, and C, what is each angle if A is twice and B three times C?

(615) Required, each angle of an isosceles triangle, if the unequal angle equals twice the sum of the other two.

(616) Construct a square whose diagonal is 3.5 inches. What is the length of its side? Ans. 2.475 in., nearly.

(617) The diagonal of a rectangle is 3 inches, the shorter side 1.5 inches; construct it and find its area.

Ans. Area = 3.897 sq. in.

(618) Through two points 1.5 inches apart, draw a circle having a diameter of 3.5 inches.

(619) At any point on a straight line, construct an angle of 30°.

(620) A line 2 inches long is met at one extremity by a second line, making with it an angle of 30°; find the center of the circle of which the first is a chord and the second a tangent.

(621) At a point on any straight line construct an angle of 45°.

(622) Construct an isosceles triangle with a base of 3 inches and a vertical angle of 90°.

(623) Draw a figure showing two parallel lines cut by a secant, and name all the angles thus formed.

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(624) What is meant by similar polygons? Give examples.

(625) One of the angles of a triangle is 30°; one of the including sides 5 inches, and the difference of the other two sides 1.5 inches; construct the triangle.

(626) How is the north end of the magnetic needle determined?

(627) How is the compass circle divided, and how are the degrees numbered?

(628) Explain why the east and west points of the compass plate are marked the reverse of their natural order.

(629) What are the principal defects of the compass?

(630) What constitutes a *course?*

(631) How is the bearing of a line taken and how checked?

(632) How is a line run by backsights?

(633) Define local attraction and state by what it is usually caused.

(634) Explain the difference between the magnetic meridian and a true meridian.

(635) What is meant by the declination of the needle?

(636) How is a true meridian established?

(637) Explain the difference between a magnetic bearing and a true bearing.

(638) (*a*) The declination is 3° 15′ east; what are the true bearings of the following lines, their magnetic bearings being:

Magnetic Bearing.	True Bearing.
N 15° 20′ E	
N 88° 50′ E	
N 20° 40′ W	
N 50° 20′ E	

(*b*) The declination is 5° 10′ west; required, the true bearing of the following lines:

Magnetic Bearing.	True Bearing.
N 7° 20′ W	
N 45° 00′ E	
S 15° 20′ E	
S 2° 30′ W	

(639) How are stations in railroad surveys numbered, and what is the interval between stations?

(640) What are the special advantages which the compass offers in preliminary railroad surveys, and what conditions should determine its rejection for such work?

(641) Describe the process of chaining.

(642) Plat the following compass notes:

Station.	Bearing.
60 + 20	
59 + 90	N 80° 15′ E
44 + 50	S 45° 55′ E
28 + 13	S 11° 25′ E
20 + 11	S 76° 30′ E
10 + 89	N 79° 25′ E
5 + 20	N 40° 50′ E
0	N 10° 10′ E

(643) Describe the vernier. A vernier is described as reading to single minutes. Make a drawing of such a vernier and explain its use.

(644) Explain the different adjustments.

(645) How is a line prolonged by backsight? by foresight?

(646) Explain by figure the process of double centering, and state its advantages.

(647) Define a horizontal angle and its measurement.

(648) Of what value is the magnetic needle where all angles are read with the vernier?

(649) Explain by example what is meant by calculated or deduced bearings.

(650) The magnetic bearing of a line is N 55° 15′ E, and an angle of 15° 17′ is turned to the right; what is the bearing of the second line?

(651) The magnetic bearing of a line is N 80° 11′ E, and an angle of 22° 13′ is turned to the right; what is the bearing of the second line?

(652) The magnetic bearing of a line is N 13° 15′ W, and an angle of 40° 20′ is turned to the left; what is the bearing of the second line?

(653)

Station.	Deflection.	Mag. Bearing.	Ded. Bearing.
54 + 25			
49 + 20	L. 25° 14′	S 25° 40′ W	
44 + 80	L. 10° 47′	S 50° 50′ W	
33 + 77	R. 16° 55′	S 61° 45′ W	
25 + 60	R. 24° 40′	S 44° 50′ W	
16 + 20	L. 15° 35′	S 20° 00′ W	
8 + 90	R. 10° 15′	S 35° 50′ W	
4 + 40	R. 15° 10′	S 25° 20′ W	
0		S 10° 15′ W	

Calculate the bearings of the foregoing transit notes of an angle line and plat them to a scale of 400 feet to the inch, showing the direction of the magnetic meridian.

(654) The line of survey *A B* (see Fig. 12) crosses a stream *C D* too wide for direct measurement. The angle *A* is 80° 20′, the angle *E* is 60° 15′, the side *A E* is 415 feet; required, the angle *B* and the side *A B*.

Ans. { Angle $B = 39° 25'$.
Side $A B = 567.44$ ft.

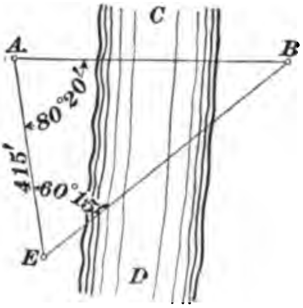

FIG. 12.

(655) An obstacle, Fig. 13, lies in the path of survey *A B*. Show how the equilateral triangle may be used in passing the object and prolonging the line of survey.

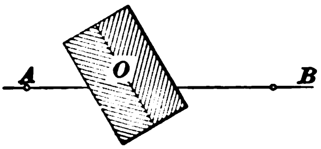

FIG. 13.

(656) What is an intersection of tangents? How is it made and the angle of intersection measured?

(657) Define *curve* and *tangent* as applied to railroad engineering.

(658) Name and describe the three classes of curves used in railroad building.

(659) What is the amount of the divergence of two lines 100 feet in length and forming an angle of 1° with each other?

(660) What is the unit curve employed in railroad building? Define it.

(661) Define a five-degree (5°) curve.

(662) What is the ratio of the degree of curve to the deflection angle for a chord of 100 feet?

(663) What is the formula for finding the *radius*, the deflection angle being given?

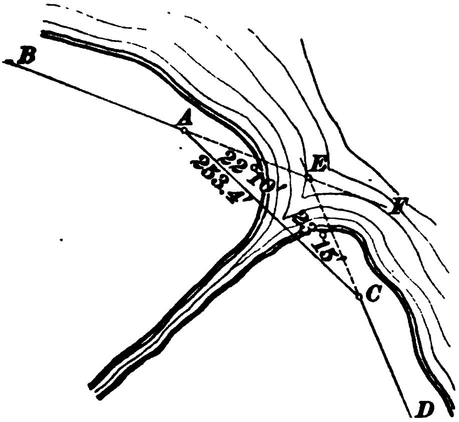

FIG. 14.

(664) What is the formula for finding the length of any chord, the radius and deflection angle being given?

(665) What is the tangent distance, and what formula is used in finding it?

(666) The tangents $A\ B$ and $C\ D$, in Fig. 14, intersect at some inaccessible point E. The angle A is 22° 10', the angle C 23° 15', and the side $A\ C$ 253.4 feet; required the angle of intersection $C\ E\ F$, and the sides $A\ E$ and $E\ C$.

Ans. { Angle $CEF = 45° 25'$.
Side $A\ E = 140.44$ ft., nearly.
Side $C\ E = 134.24$ ft., nearly.

(667) The angle of intersection of two tangents is 35° 10'; the degree of curve is 6° 15'; what is the tangent distance?

Ans. 290.66 ft., nearly.

(668) The angle of intersection is 14° 12'; the degree of curve, 3° 15'; what is the tangent distance?

Ans. 219.62 feet.

(669) How is the length of curve found?

(670) The angle of intersection is 30° 45'; the degree of curve, 5° 15'; what is the length of the curve?

Ans. 585.71 feet.

(671) The angle of intersection is 33° 06'; the station of the point of intersection, 20 + 37.8; the degree of curve, 5°; what is the station of the P. C., length of curve and station of the P. T.?

Ans. { P. C. = 16 + 97.17.
Length of curve = 662 ft.
P. T. = 23 + 59.17.

(672) The angle of intersection is 20° 10′; the tangent distance, 291.16 feet; required, the radius and degree of curve.

Ans. { Radius = 1,637.29 feet.
Degree of curve, 3° 30′.

(673) What is the formula for the chord deflection ?

(674) What is the ratio of the chord deflection to the tangent deflection ?

(675) The degree of curve is 7°; what is the deflection angle for a chord of 48.2 feet ? Ans. 1° 41.22′.

(676) The degree of curve is 6° 15′; what is the deflection angle for a chord of 72.7 feet ? Ans. 2° 16.3′.

(677) The degree of curve is 5° 30′; what is the tangent deflection, or offset, for 50 feet ? Ans. 1.199 feet.

(678) The degree of curve is 4° 15′; what is the chord deflection for 35.2 feet ? Ans. 0.919 foot.

(679) What is the radius of a 3° 10′ curve ?

Ans. 1,809.57 feet.

(680) Two lines of equal length forming an angle of 1° with each other diverge 18.22 feet; what are the lengths of the lines ?

Ans. { By trigonometry, 1,043.53 feet.
By practical method, 1,044.13 feet.

(681) A curve is 606.25 feet in length; the angle of intersection is 24° 15′; what is the degree of the curve ?

Ans. 4°.

(682) What are the different processes of leveling ? Define them.

(683) Describe the **Y** level and explain its adjustments.

(684) The level rod is held 300 feet from the instrument; the reading is 6.81 feet. After causing the level bubble to move over one division of the scale, the reading is 6.84; what is the radius of the bubble tube ? Ans. 100 feet.

(685) What is meant by the *power* and *definition* of a telescope ?

(686) Describe the self-reading leveling rod.

(687) The elevation of the point where the backsight is taken is 61.84 feet, the backsight is 11.81 feet, and the foresight to a turning point (T. P.) is 0.49 foot; what is the elevation of the T. P.? Ans. 73.16 feet.

(688) Define a datum line.

(689) What are turning points?

(690) What are bench marks?

(691) What are the principal sources of error in taking levels?

(692) Explain the process of checking level notes.

(693) What is a profile?

(694) Define a 1 per cent. grade.

(695) The elevation of the grade at Station 66 is 126.50; between Stations 66 and 100 there is an ascending grade of 1.25 per cent.; what is the elevation of the grade at Station 93? Ans. 160.25 feet.

(696) What is topographical surveying?

(697) The elevation of a station is 56.5 feet; the ground on the left falls 10.3 feet in a distance of 73 feet, where the slope changes, giving a fall of 16.4 feet in a distance of 56 feet. On the right the ground rises 11.4 feet in a distance of 84 feet, where the slope changes, giving a rise of 8.8 feet in a distance of 96 feet. How are these slopes recorded in the topographer's book? How many 10-foot contours are included by the side slopes, and what are their elevations, the contours being placed at even decimal intervals of 10 feet? What are the several distances of the contours from the center line?

(698) Work out the elevation of the following notes; check the notes, plat them in a profile, and draw a descending grade line of 80 feet to the mile, and place in the

column headed Grade the elevation of the grade of each station given in the station column:

Station.	Rod Reading.	Ht. Instrument.	Elevation.	Grade.	Remarks.
B. M.	+5.53		161.42		B. M. on Poplar tree 10 ft. left; Sta. 40.
40	6.4			162.0	
41	7.2				
41 + 60	10.9				
42	8.6				
43	8.8				
T. P. –	8.66				
+	2.22				
44	4.8				
45	6.3				
46	8.8				
47	9.9				
48	11.1				
T. P. –	11.24				
+	3.30				
49	4.7				
50	7.1				
51	8.7				
52	9.8				
53	10.9				
T. P. –	11.62				

(699) The ground to the right of the center line has an ascending slope of 11° for a distance of 120 feet, and to

the left of the center line a descending slope of 9° for a distance of 117 feet, where the slope changes to a descending slope of 5° for a distance of 65 feet; how are the slopes recorded in the topographer's book? If the elevation of the ground at the center line is 75.0 feet, how many 5-foot contours are included by the given slopes, and what are their respective distances from the center line?

(700) The angle of slope is 3°; what is the horizontal distance for a rise of 10 feet?

(701) An instrument is stationed at *A*, Fig. 15, 100 feet from the base of a church spire *B C*. The horizontal line of sight *A C* from the instrument is 5 feet above the base of the spire, which at that point is 30 feet in diameter. The angle *C A B* is 45° 20′; required, the height of the spire. Ans. 121.345 feet.

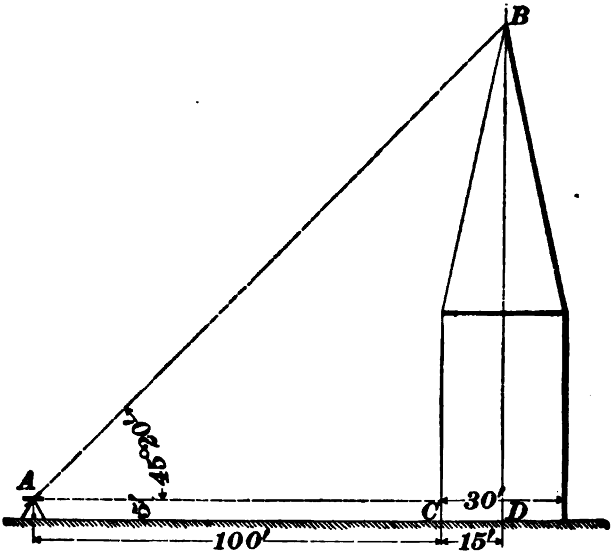

FIG. 15.

(702) The height of barometer *h* at lower station is

29.40 in.; the temperature $t = 74°$; the height of barometer H at the higher station is 26.95 in., and the temperature $t' = 58°$; what is the difference in elevation?

Asn. 2,454 feet.

(703) What is hydrographic surveying?

(704) What stage of tide is the basis for all soundings?

(705) What is a tide gauge?

LAND SURVEYING.

(ARTS. 1309–1344.)

EXAMINATION QUESTIONS.

(706) What are the names of the general divisions into which the public lands of the United States are divided?

(707) Give the dimensions and contents of each.

(708) (*a*) Define a *Principal Meridian.* (*b*) How is it established? (*c*) What boundaries are marked on it?

(709) What causes *convergency* of *meridians*, and what effect does such convergency have upon the boundary lines of the Government surveys?

(710) (*a*) What are *Standard Parallels?* (*b*) What is their object? (*c*) What boundaries are marked on them?

(711) How are townships described with reference to *Base Line* and *Principal Meridian?* Illustrate by a diagram.

(712) Show by a diagram how township lines are run and the order of the survey.

(713) Explain by figure the terms *random line* and *true line.*

(714) How are excesses or deficiencies of measurement disposed of in townshiping and sectioning?

(715) Show by a diagram the subdivisions of townships and the order of survey.

(716) (*a*) Define *meandering* as applied to Government surveys. (*b*) In mapping, what use is made of meander lines? (*c*) How are islands located?

(717) What is a *line tree*, how is it marked, and how are trees on either side of the line of survey marked?

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(718) How is a boundary corner in a timbered country marked?

(719) Describe the *township* corner-post or corner-stone. How is it set, and how marked?

(720) Describe a *section* corner, and explain how it is set and marked.

(721) How are *mound* corners constructed?

(722) What are *double corners*, where are they found, and how are they designated?

(723) In running township and section lines, what account should the surveyor take of the topography of the country?

(724) What are the various field books used in the survey of Government lands?

(725) State why the retracing of *old lines* or original surveys is so difficult.

(726) Suppose the original notes of a farm survey are the following, and suppose that only two of the original corners, viz., *B* and *C*, remain, and that we find the present bearing from *B* to *C* is N 60° 15′ E, how may we determine the magnetic variation, and what will be the corrected bearings by which we may restore the original boundaries?

Stations.	Bearings.	Distances.
A	N 31½° W	10.4 chains.
B	N 62° E	9.2 chains.
C	S 36° E	7.6 chains.
D	S 45½° W	10.0 chains.

(727) How are boundary lines straightened and new corners established where old ones are obliterated?

(728) What are witness trees, and how are they marked?

(729) Explain by a figure the triangular method of calculating areas.

(730) Explain by a figure the trapezoidal method of calculating areas.

(731) Define (*a*) the latitude of a point; (*b*) the longi-

tude of a point; (*c*) the latitude of a line, and (*d*) the departure of a line.

(732) The line *A B*, in Fig. 16, has a bearing of N 30° E and a length of 187 ft. Complete the figure, showing the latitude and departure of *A B*, and calculate their respective lengths by means of a table of sines and cosines.

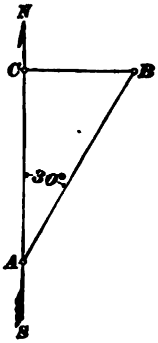

FIG. 16.

(733) Describe a *traverse table.*

(734) The bearing of a line is N 23¼° E; its length is 423 ft.; calculate its latitude and departure by the use of a traverse table, and explain the process.

(735) The length of a line is 225 ft.; its bearing is 40°; what are the latitude and departure of the line, and what is the relation of that latitude and departure to the latitude and departure of the complement of the given bearing?

(736) How are latitudes and departures applied in testing the accuracy of a survey?

(737)

Stations.	Bearings.	Distances.
1	N 31½° W	10.40 chains.
2	N 62° E	9.20 chains.
3	S 36° E	7.60 chains.
4	S 45½° W	10.00 chains.

Calculate the latitudes and departures of the courses in the accompanying example, and balance them, giving all the steps of the process.

(738) Calculate the total latitudes and departures from Station 2, and from them make a plat of the survey explaining the different steps in the process.

(739) What is the *longitude* of a line?

(740) What is the general rule for the *double longitude* of any course of a given survey?

(741) Illustrate by a figure the method of computing areas by double longitudes.

(742) What are *north products* and what *south products?*

(743) If the double longitude is negative and the corresponding latitude positive, is the product *north* or *south?*

(744) How is the most easterly or most westerly station of a survey found?

(745)

Stations.	Bearings.	Distances.
1	S 21° W	12.41 chains.
2	N 83¼° E	5.86 chains.
3	N 12° E	8.25 chains.
4	N 47° W	4.24 chains.

Find the area by double longitudes, giving all the intermediate steps.

Ans. 4 A. 2 R. 35.8 P.

(746)

Stations.	Bearings.	Distances.
1	S 21¼° W	17.62 chains.
2	S 34° W	10.00 chains.
3	N 56° W	14.15 chains.
4	N 34° E	9.76 chains.
5	N 67° E	2.30 chains.
6	N 23° E	7.03 chains.
7	N 18½° E	4.43 chains.
8	S 76½° E	12.41 chains.

Find area by double longitudes.

Ans. 33 A. 0 R. 8.8 P.

(747)

Stations.	Bearings.	Distances
1	N 18¾° E	1.93 chains.
2	N 9° W	1.29 chains.
3	N 14° W	2.71 chains.
4	N 74° E	0.95 chains.
5	S 48½° E	1.59 chains.
6	S 14¼° E	1.14 chains.
7	S 19½° E	2.15 chains.
8	S 23½° W	1.22 chains.
9	S 5° W	1.40 chains.
10	S 30° W	1.02 chains.
11	S 81½° W	0.69 chains.
12	N 32½° W	1.98 chains.

Find area by double longitudes.

Ans. 1 A. 1 R. 27¾ P.

(748) In laying out town sites, what matters should first be considered?

(749) (*a*) How should the grades of streets be arranged with reference to drainage? (*b*) with reference to topography?

(750) How should *base lines* be located with reference to railroad lines?

(751) What is the usual order of preliminary survey?

(752) How are measurements to be made?

(753) Give a brief sketch illustrating the mode of laying out *base lines* and *subdivisions.*

(754) How are *base lines* rendered permanent?

(755) What important requirement is met by locating street corners by intersecting lines?

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